Journal of Quality Measurement and Analysis *JQMA* 12(1-2) 2016, 53-65
Jurnal Pengukuran Kualiti dan Analisis

ON CERTAIN CLASSES OF MEROMORPHIC HARMONIC CONCAVE FUNCTIONS DEFINED BY AL-OBOUDI OPERATOR

(Berkenaan Kelas Fungsi Cekung Meromorfi Harmonik Tertentu yang Ditakrif oleh Pengoperasi Al-Oboudi)

KHALID CHALLAB & MASLINA DARUS

ABSTRACT

In this work, a class of meromorphic harmonic concave functions defined by Oboudi operator in the punctured unit disc is introduced. Coefficient conditions, distortion inequalities, extreme points, convolution bounds, geometric convolution, integral convolution and convex combinations for functions f belonging to this class are obtained.

Keywords: Meromorphic function; harmonic function; concave function; Al-Oboudi operator

ABSTRAK

Dalam kajian ini, kelas fungsi cekung meromorfi harmonik yang ditakrif oleh pengoperasi Al-Oboudi diperkenalkan dalam cakera unit terpancit. Syarat pekali, ketaksamaan erotan, titik ekstrim, batas konvolusi, konvolusi geometri, konvolusi kamiran dan gabungan cembung untuk fungsi *f* dalam kelas tersebut diperoleh.

Kata kunci: fungsi meromorfi; fungsi harmonik; fungsi cekung; pengoperasi Al-Oboudi

1. Introduction

Conformal maps of the unit disc onto convex domain are very classic. We note that Avkhadiev and Wirths (2005) discovered the conformal mapping of a unit disc onto concave domains (the complements of convex closed sets). It is quite interesting to see other related results in this direction as not many problems are discussed thoroughly towards this approach.

Let $U = \{z \in \mathbb{C}: |z| < 1\}$ denote the open unit disc, where f has the form given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that maps U conformally onto a domain whose complement with respect to C is convex and that satisfies the normalisation $f(1)=\infty$. In addtion, they imposed on these functions the condition that the opening angle of $f(U)$ at infinity is less than or equal to $\alpha\pi, \alpha \in (1,2]$. These families of functions are denoted by $C_0(\alpha)$. The class $C_0(\alpha)$ is referred to as the class of concave univalent functions.

Chuaqui *et al*. (2012) defined the concept of meromorphic concave mappings. A conformal mapping of meromorphic function in U^*, where $U^* = U\backslash\{0\}$ is said to be a concave mapping if its image is the complement of a compact convex set.
If f has the form

$$f(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \ldots,$$

then a necessary and sufficient condition for f to be a concave mapping is

$$1+Re\left\{z\frac{f''(z)}{f'(z)}\right\}<0,\quad |z|<1.$$

A continuous function $f=u+iv$ is a complex valued harmonic function in a domain $U\subset\mathbb{C}$ if both u and v are real harmonic in U. In any simply connected domain, we write $f=h+\bar{g}$ where h and g are analytic in U. A necessary and sufficient condition for f to be locally univalent and orientation preserving in U is that $|h'|>|g'|$ in U (see Clunie and Sheil-Small 1984). Hengartner and Schober (1987) investigated functions harmonic in the exterior of the unit disc $U=\{z\in\mathbb{C}:|z|<1\}$. They showed that complex valued, harmonic, sense preserving, univalent mapping f must admit the representation.

$$f(z)=h(z)+\overline{g(z)}+Alog|z|,$$

where $h(z)$ and $g(z)$ are defined by

$$h(z)=\alpha z+\sum_{n=1}^{\infty}a_n z^{-n},\quad g(z)=\beta\bar{z}+\sum_{n=1}^{\infty}b_n z^{-n}$$

for $0\le|\beta|<|\alpha|$ and $A\in\mathbb{C}$, $z\in U^{**}$,where $U^{**}=\{z\in\mathbb{C}:|z|>1\}$.

We call h the analytic part and g the co-analytic part of f.
For $z\in U^{*}$, let M_H be the class of functions:

$$f(z)=h(z)+\overline{g(z)}=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^k+\sum_{k=1}^{\infty}\overline{b_k z^k} \tag{2}$$

which are analytic in the punctured unit disc U^{*}, where $h(z)$ and $g(z)$ are analytic in U and U^{*}, respectively, and $h(z)$ has a simple pole at the origin with residue 1 (see Al-Shaqsi and Darus 2008).

A function $f\in M_H$ is said to be in the subclass MS_H^{*} of meromorphically harmonic starlike functions in U^{*} if it satisfies the condition

$$Re\left\{-\frac{zh'(z)-\overline{zg'(z)}}{h(z)+g(z)}\right\}>0,\qquad \left(z\in U^{*}\right).$$

Also, a function $f\in M_H$ is said to be in the subclass MC_H of meromorphically harmonic convex functions in U^{*} if it satisfies the condition

$$Re\left\{-\frac{zh''(z)+h'(z)-\overline{zg''(z)+g'(z)}}{h'(z)+g'(z)}\right\}>0,\quad \left(z\in U^{*}\right).$$

Note that the classes of harmonic meromorphic starlike functions, harmonic meromorphic convex functions and harmonic meromorphic concave functions (MHC_0) have been studied by Jahangiri and Silverman (1999), Jahangiri (2000), Jahangiri (1998), Aldawish and Darus (2015) and Challab and Darus (2016). Other works related to concave can be read in Aldawish and Darus (2015), Darus *et al.* (2015) and Aldawish *et al.* (2014.)

AL-Oboudi (2004) introduced the operator D^n for $f\in A$ which is the class of function of the form (1) analytic in the unit disc $U=\{z\in\mathbb{C}:|z|<1\}$, and defined the following differential operator

$$D^0 f(z)=f(z),$$

$$D^1 f(z) = (1-\gamma) + \gamma z f'(z) = D_\gamma f(z), \gamma \geq 0, \tag{3}$$

$$D^n f(z) = D_\gamma \left(D^{n-1} f(z) \right). \tag{4}$$

If f is given by (1), then from (3) and (4) we see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\gamma]^n a_k z^k. \tag{5}$$

Al-Oboudi (2004) introduced the operator O^n for $f \in MHC_0$ which is the class of functions $f = h + g$ that are harmonic univalent and sense-preserving in the unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$ for which $\{ f(0) = h(0) = f_z(0) - 1 = 0 \}$.
Now, we define O^n for $f = h + \overline{g}$ given by (2) as

$$\left\{ O^n f(n) = O^n h(n) + \overline{O^n g(n)} \right\}, n = 0,1,2,..., z \in U^*,$$

where

$$O^n h(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} \left[1 + (k-1)\gamma \right]^n a_k z^k,$$

$$O^n \overline{\mathrm{g(z)}} = \sum_{k=1}^{\infty} \left[1 + (k-1)\gamma \right]^n \overline{b_k z^k}.$$

This work is an attempt to give a connection between harmonic function and meromorphic concave functions defined by Oboudi operator by introducing a class $O^n MHC_0$ of meromorphic harmonic concave functions.

Definition 1.1. For $n \in N_0 = \{1, 2, ...\}$, $\gamma \in [1, 2]$, $k \geq 1$, let $O^n MHC_0$ denote the class of meromorphic harmonic concave functions $O^n f(z)$ defined by Oboudi differential operator of the form,

$$O^n f(z) = (-1)^n / z + \sum_{k=1}^{\infty} [1 + (k-1)\gamma]^n a_k z^k + \sum_{k=1}^{\infty} [1 + (k-1)\gamma]^n (\overline{b_k z^k}) \tag{6}$$

such that

$$1 + Re \left\{ \frac{z \left(O^n f(z) \right)''}{(O^n f(z))'} \right\} < 0.$$

2. Coefficient Conditions

In this section, sufficient condition for a function $O^n f(z) \in O^n MHC_0$ is derived.

Theorem 2.1. *Let* $O^n f(z) = h + g'$ *be of the form*

$$O^n f(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n a_k z^k + \sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n \overline{b_k z^k}.$$

If $\sum_{k=1}^{\infty} k^2\left[1+(k-1)\gamma\right]^n\left(|a_k|+|b_k|\right) \le 1,$

then $O^n f(z)$ *is harmonic univalent, sense preserving in* U^{**}.

Proof: First for $0 < |z_1| \le |z_2| < 1$, we have

$$|O^n f(z_1) - O^n f(z_2)| = |(-1)^n / z_1 + \sum_{k=1}^{\infty}[1+(k-1)\gamma]^n a_k z_1^{\,k} + \sum_{k=1}^{\infty}[1+(k-1)\gamma]^n \overline{(b_k z_1^{\,k})}$$

$$-(-1)^n / z_2 - \sum_{k=1}^{\infty}[1+(k-1)\gamma]^n a_k z_2^{\,k} - \sum_{k=1}^{\infty}[1+(k-1)\gamma]^n \overline{(b_k z_2^{\,k})}\,|$$

$$\ge 1/|z_1| - 1/|z_2|$$

$$-\sum_{k=1}^{\infty}[1+(k-1)\gamma]^n\,|a_k|\,|z_1^{\,k} - z_2^{\,k}| - \sum_{k=1}^{\infty}[1+(k-1)\gamma]^n\,|b_k|\,|z_1^{\,k} - z_2^{\,k}|$$

$$> \frac{|z_1 - z_2|}{|z_1||z_2|} - |z_1 - z_2|\sum_{k=1}^{\infty} k\left[1+(k-1)\gamma\right]^n\left(|a_k|+|b_k|\right)$$

$$> |z_1 - z_2| / |z_1||z_2|\,[1 - |z_2|^2 \sum_{k=1}^{\infty} k^2[1+(k-1)\gamma]^n(|a_k|+|b_k|)]$$

$$> |z_1 - z_2| / |z_1||z_2|\,[1 - \sum_{k=1}^{\infty} k^2[1+(k-1)\gamma]^n(|a_k|+|b_k|)].$$

The last expression is non negative by $\sum_{k=1}^{\infty} k^2[1+(k-1)\gamma]^n(|a_k|+|b_k|) \le 1$ and $O^n f(z)$ is univalent in U^*.

Now we want to show f is sense preserving in U^{**}, we need to show that $|h'(z)| \ge |g'(z)|$ in U^*.

$$|h'(z)| \ge \frac{1}{|z|^2} - \sum_{k=1}^{\infty} k\left[1+(k-1)\gamma\right]^n |a_k||z|^{k-1}$$

$$= \frac{1}{r^2} - \sum_{k=1}^{\infty} k\left[1+(k-1)\gamma\right]^n |a_k| r^{k-1}$$

$$>1-\sum_{k=1}^{\infty}k\left[1+(k-1)\gamma\right]^n|a_k|$$

$$\geq 1-\sum_{k=1}^{\infty}k^2\left[1+(k-1)\gamma\right]^n|a_k|$$

$$\geq \sum_{k=1}^{\infty}k^2\left[1+(k-1)\gamma\right]^n|b_k|$$

$$>\sum_{k=1}^{\infty}k\left[1+(k-1)\gamma\right]^n|b_k|r^{k-1}=\sum_{k=1}^{\infty}k\left[1+(k-1)\gamma\right]^n|b_k||z|^{k-1}\geq\left|g^{'}(z)\right|.$$

Thus this completes the proof of the Theorem 2.1. □

Theorem 2.2. *Let* $O^n f(z)=h+\overline{g}$ *be of the form,*

$$O^n f(z)=\frac{(-1)^n}{z}+\sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n a_k z^k+\sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n \overline{b_k z^k}.$$

Then $O^n f(z)\in O^n MHC_{\circ}$ *if the inequality*

$$\sum_{k=1}^{\infty}k^2[1+(k-1)\gamma]^n(|a_k|+|b_k|)\leq 1 \tag{7}$$

holds for coefficient $O^n f(z)=h+g$.

Proof: Using the fact that $Re(w)<0\leftrightarrow\left|\frac{w+1}{w-1}\right|<1$, it suffices to show

$$\left|\frac{w+1}{w-1}\right|<1.$$

Let $w=1+Re\left\{\frac{z\left(O^n f(z)\right)^{''}}{(O^n f(z))^{'}}\right\}$, such that

$w=\frac{zg^{'}(z)}{g(z)}$, where $g(z)=z(O^n f(z))'$.

Now,

$$\left|\frac{w+1}{w-1}\right| = \left| \frac{\sum_{k=1}^{\infty}(k^2+k)\left[1+(k-1)\gamma\right]^n a_k z^k - \sum_{k=1}^{\infty}(k^2+k)\left[1+(k-1)\gamma\right]^n \overline{b_k z^k}}{\frac{-2(-1)^{n+1}}{z} + \sum_{k=1}^{\infty}(k^2-k)\left[1+(k-1)\gamma\right]^n a_k z^k - \sum_{k=1}^{\infty}(k^2-k)\left[1+(k-1)\gamma\right]^n \overline{b_k z^k}} \right|$$

$$\left|\frac{w+1}{w-1}\right| < \frac{\sum_{k=1}^{\infty}(k^2+k)\left[1+(k-1)\gamma\right]^n |a_k| + \sum_{k=1}^{\infty}(k^2+k)\left[1+(k-1)\gamma\right]^n |b_k|}{2-\sum_{k=1}^{\infty}(k^2-k)\left[1+(k-1)\gamma\right]^n |a_k| - \sum_{k=1}^{\infty}(k^2-k)\left[1+(k-1)\gamma\right]^n |b_k|}. \tag{8}$$

The last expression is bounded above by 1 if

$$\sum_{k=1}^{\infty}(k^2+k)\left[1+(k-1)\gamma\right]^n |a_k| + \sum_{k=1}^{\infty}(k^2+k)\left[1+(k-1)\gamma\right]^n |b_k|$$

$$\le 2-\sum_{k=1}^{\infty}(k^2-k)[1+(k-1)\gamma]^n\,|a_k| - \sum_{k=1}^{\infty}(k^2-k)[1+(k-1)\gamma]^n\,|b_k|$$

which is equivalent to our condition by

$$\sum_{k=1}^{\infty}k^2[1+(k-1)\gamma]^n(|a_k|+|b_k|)\le 1. \qquad \square$$

Theorem 2.3. *Let* $O^n f(z) = h+g$ *be of the form*

$$O^n f(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n a_k z^k + \sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n \overline{b_k z^k}.$$

A necessary and sufficient condition for $O^n f(z)$ *to be in* $O^n MHC_{\circ}$ *is that*

$$\sum_{k=1}^{\infty}k^2\left[1+(k-1)\gamma\right]^n\left(|a_k|+|b_k|\right)\le 1.$$

Proof: In view of Theorem 2.2, we assume that $\sum_{k=1}^{\infty}k^2[1+(k-1)\gamma]^n(|a_k|+|b_k|)>1.$

Since $O^n f(z)\in O^n MHC_0$, then

$1+Re\{(z(O^n f(z))'')/(O^n f(z))'\}$ equivalent to

$$Re\frac{zg'(z)}{g(z)} = Re\frac{z\left(\frac{(-1)^{n+2}}{z^2} + \sum_{k=1}^{\infty}k^2\left[1+(k-1)\gamma\right]^n a_k z^{k-1} + \sum_{k=1}^{\infty}k^2\left[1+(k-1)\gamma\right]^n \overline{b_k z^{k-1}}\right)}{\frac{(-1)^{n+1}}{z} + \sum_{k=1}^{\infty}k\left[1+(k-1)\gamma\right]^n a_k z^k + \sum_{k=1}^{\infty}k\left[1+(k-1)\gamma\right]^n \overline{b_k z^k}}$$

$$= Re\frac{\frac{-(-1)^{n+1}}{z}+\sum_{k=1}^{\infty}k^2\left[1+(k-1)\gamma\right]^n a_k z^{k-1}+\sum_{k=1}^{\infty}k^2\left[1+(k-1)\gamma\right]^n \overline{b_k z^{k-1}}}{\frac{(-1)^{n+1}}{z}+\sum_{k=1}^{\infty}k\left[1+(k-1)\gamma\right]^n a_k z^{k}+\sum_{k=1}^{\infty}k\left[1+(k-1)\gamma\right]^n \overline{b_k z^{k}}}\le 0$$

for $|z| = r > 1$, the above expression reduce to

$$Re\left(\frac{-(-1)^{n+1}+\sum_{k=1}^{\infty}k^2\left[1+(k-1)\gamma\right]^n(|a_k|+|b_k|)r^k}{(-1)^{n+1}+\sum_{k=1}^{\infty}k\left[1+(k-1)\gamma\right]^n(|a_k|+|b_k|)r^k}\right)=\left(\frac{A(r)}{B(r)}\right)\le 0.$$

As we assumed $\sum_{k=1}^{\infty}k^2[1+(k-1)\gamma]^n(|a_k|+|b_k|)>1$, then A(r) and B(r) are positive for r sufficiently close to 1. Thus there exists a $z_{\circ} = r_{\circ} > 1$ for which the quotient is positive. This contradicts the required condition that $\frac{A(r)}{B(r)}\le 0$, so the proof is complete. □

3. Distortion bounds and extreme points

Bounds and extreme points for functions f belonging to the class O^nMHC are estimated in this section.

Theorem 3.1. *If* $O^n f_k = O^n h_k + O^n g_k \in O^n MHC$ *and* $0<|z|=r<1$ *then* $\left|O^n f_k(z)\right|\le\frac{1+r^2}{r}$ *and* $\left|O^n f_k(z)\right|\ge\frac{1-r^2}{r}$.

Proof: Let $O^n f_k = O^n h_k + O^n g_k \in O^n MHC_0$. Taking the absolute value of $O^n f_k$, we obtain

$$\left|O^n f_k(z)\right|=\left|\frac{(-1)^n}{z}+\sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n a_k z^k+\sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n\overline{b_k z^k}\right|$$

$$\left|O^n f_k(z)\right|\ge\frac{1}{r}-\sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n\left(|a_k|+|b_k|\right)r^k$$

$$\ge\frac{1}{r}-\sum_{k=1}^{\infty}k^2\left[1+(k-1)\gamma\right]^n\left(|a_k|+|b_k|\right)r^k$$

$$\ge\frac{1}{r}-\sum_{k=1}^{\infty}k^2\left[1+(k-1)\gamma\right]^n\left(|a_k|+|b_k|\right)r$$

by applying

$$\sum_{k=1}^{\infty}k^2[1+(k-1)\gamma]^n(|a_k|+|b_k|)\le 1,$$

then

$$\left|O^n f_k(z)\right| \ge \frac{1}{r} - r = \frac{1-r^2}{r}.$$

Now,

$$\left|O^n f_k(z)\right| = \left|\frac{(-1)^n}{z} + \sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n a_k z^k + \sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n \overline{b_k z^k}\right|$$

$$\le \frac{1}{r} + \sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n \left(|a_k|+|b_k|\right) r^k$$

$$\le \frac{1}{r} + \sum_{k=1}^{\infty} k^2 \left[1+(k-1)\gamma\right]^n \left(|a_k|+|b_k|\right) r.$$

By applying $\sum_{k=1}^{\infty} k^2[1+(k-1)\gamma]^n(|a_k|+|b_k|) \le 1$, then

$$\left|O^n f_k(z)\right| \le \frac{1}{r} + r = \frac{1+r^2}{r}. \qquad \square$$

Theorem 3.2. *Let* $O^n f_n = O^n h_n + O^n g_n$ *and* $O^n f_n(z) = O^n h_n(z) + O^n g_n(z)$

That is

$$O^n f_n(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n a_k z^k + \sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n \overline{b_k z^k}.$$

Set

$$O^n h_{n,0} = O^n g_{n,0} = \frac{(-1)^n}{z} \quad \textit{and} \quad O^n h_{n,k}(z) = \frac{(-1)^n}{z} + \frac{1}{k^2} z^k,$$

For $k = 1,2,3,...$ *and let*

$$O^n g_{n,k}(z) = \frac{(-1)^n}{z} + \frac{1}{k^2}\overline{z^k},$$

for $k = 1,2,3,...$ *then* $O^n f_n \in O^n MHC$ *if and only if* $O^n f_n$ *can be expressed as*

$$O^n f_{n,k} = \sum_{k=0}^{\infty}\left(\lambda_k O^n h_{n,k}(z) + \gamma_k O^n g_{n,k}(z)\right),$$

where $\lambda_k \ge 0, \gamma_k \ge 0$ *and* $\sum_{k=0}^{\infty}(\lambda_k + \gamma_k) = 1$. *In particular the extreme points of* $O^n MHC_\circ$ *are* $\left\{O^n h_{n,k}\right\}$ *and* $\{O^n g_{n,k}\}$.

Proof: For functions $O^n f_n = O^n h_n + O^n \overline{g}_n$, where $O^n h_n$ and $O^n \overline{g}_n$ are given by

$$O^n f_n(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} \left[1+(k-1)\gamma\right]^n a_k z^k + \sum_{k=1}^{\infty} \left[1+(k-1)\gamma\right]^n \overline{b_k z^k},$$

we have

$$O^n f_{n,k}(z) = \sum_{k=0}^{\infty} \left(\lambda_k O^n h_{n,k}(z) + \gamma_k O^n g_{n,k}(z)\right)$$

$$= \lambda_0 O^n h_{n,0} + \gamma_0 O^n g_{n,0} + \sum_{k=1}^{\infty} \left(\lambda_k O^n h_{n,k}(z) + \gamma_k O^n g_{n,k}(z)\right)$$

$$= \lambda_0 O^n h_{n,0} + \gamma_0 O^n g_{n,0} + \sum_{k=1}^{\infty} \lambda_k \left(\frac{(-1)^n}{z} + \frac{1}{k^2} z^k\right) + \sum_{k=1}^{\infty} \gamma_k \left(\frac{(-1)^n}{z} + \frac{1}{k^2} \overline{z^k}\right)$$

$$= \sum_{k=0}^{\infty} (\lambda_k + \gamma_k) \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\lambda_k z^k + \gamma_k \overline{z^k}\right).$$

Now by Theorem 2.2,

$$\sum_{k=1}^{\infty} \lambda_k \frac{1}{k^2} k^2 + \gamma_k \frac{1}{k^2} k^2 = \sum_{k=1}^{\infty} \lambda_k + \gamma_k = 1 - \lambda_0 - \gamma_0 \leq 1,$$

We have $O^n f_{(n,k)(z)} \in O^n MHC_0$.

Conversely, suppose that $O^n f_{n,k}(z) \in O^n MHC_\circ$. Setting

$$\lambda_k = k^2 \left|\left[1+(k-1)\gamma\right]^n a_k\right|,\ k \geq 1$$

and

$$\gamma_k = k^2\, |[1+(k-1)\gamma]^n b_k|, k \geq 1.$$

We define

$$\lambda_0 + \gamma_0 = 1 - \sum_{k=1}^{\infty} \lambda_k - \sum_{k=1}^{\infty} \gamma_k.$$

Therefore $O^n f_n(z)$ can be written as

$$O^n f_n(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} \left|\left[1+(k-1)\gamma\right]^n a_k\right| z^k + \sum_{k=1}^{\infty} \left|\left[1+(k-1)\gamma\right]^n b_k\right| \overline{z^k}$$

$$=\frac{(-1)^n}{z}+\sum_{k=1}^{\infty}\lambda_k\frac{1}{k^2}z^k+\gamma_k\frac{1}{k^2}z^k$$

$$=\frac{(-1)^n}{z}+\sum_{k=1}^{\infty}\left(O^n h_{n,k}(z)-\frac{(-1)^n}{z}\right)\lambda_k+\sum_{k=1}^{\infty}\left(O^n \mathrm{g}_{n,k}(z)-\frac{(-1)^n}{z}\right)\gamma_k$$

$$=\frac{(-1)^n}{z}\left(1-\sum_{k=1}^{\infty}\lambda_k-\sum_{k=1}^{\infty}\gamma_k\right)+\sum_{k=1}^{\infty}O^n h_{n,k}(z)\lambda_k+\sum_{k=1}^{\infty}O^n \mathrm{g}_{n,k}(z)\gamma_k$$

$$=(\lambda_0+\gamma_0)\frac{(-1)^n}{z}+\sum_{k=1}^{\infty}O^n h_{n,k}(z)\lambda_k+\sum_{k=1}^{\infty}O^n \mathrm{g}_{n,k}(z)\gamma_k$$

and finally

$$O^n f_n(z)=\sum_{k=0}^{\infty}\left(\lambda_k O^n h_{n,k}(z)+\gamma_k O^n \mathrm{g}_{n,k}(z)\right).$$

The proof is completed, therefore $,O^n h_{(n,k)(z)}$ and $O^n g_{(n,k)(z)}$ are extreme points. □.

4. Convolution Properties

In this section, we define and study the convolution, geometric convolution and integral convolution of the class $O^n MHC_{\circ}$.

For harmonic function $O^n f_n, O^n F_n$ is defined as follows:

$$O^n f_n(z)=\frac{(-1)^n}{z}+\sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n|a_k|z^k+\sum_{k=1}^{\infty}\left[1+(k-1)\gamma\right]^n|b_k|\overline{z^k} \tag{9}$$

and

$$O^n F_n(z)=(-1)^n/z+\sum_{k=1}^{\infty}[1+(k-1)\gamma]^n|A_k|z^k+\sum_{k=1}^{\infty}[1+(k-1)\gamma]^n|B_k|\overline{z^k}. \tag{10}$$

The convolution of $O^n f_n$ and $O^n F_n$ is given by

$$(O^n f_n * O^n F_n)(z)=O^n f_n(z)*O^n F_n(z)$$

$$=(-1)^n/z+\sum_{k=1}^{\infty}[1+(k-1)\gamma]^n|a_k||A_k|z^k+\sum_{k=1}^{\infty}[1+(k-1)\gamma]^n|b_k||B_k|\overline{z^k}.$$

The geometric convolution $O^n f_n$ and $O^n F_n$ is given by

$$(O^n f_n \bullet O^n F_n)(z)=O^n f_n(z)\bullet O^n F_n(z)$$

$$=(-1)^n/z+\sum_{k=1}^{\infty}[1+(k-1)\gamma]^n\sqrt{|a_k A_k|}z^k+\sum_{k=1}^{\infty}[1+(k-1)\gamma]^n\sqrt{|b_k B_k|}\overline{z^k}.$$

The integral convolution of $O^n f_n$ and $O^n F_n$ is given by

$$(O^n f_n \cdot O^n F_n)(z) = O^n f_n(z) \cdot O^n F_n(z)$$
$$= (-1)^n / z + \sum_{k=1}^{\infty}[1+(k-1)\gamma]^n \, |a_k A_k| / kz^k + \sum_{k=1}^{\infty}[1+(k-1)\gamma]^n \, |b_k B_k| / k\overline{z^k}.$$

Theorem 4.1. *Let* $O^n f_n \in O^n MHC_0$ *and* $O^n F_n \in O^n MHC_0$. *Then the convolution* $O^n f_n * O^n F_n \in O^n MHC_0$.

Proof: For $O^n f_n$ and $O^n F_n$ given by (9) and (10), then the convolution is given by (10). We show that the coefficients of $O^n f_n * O^n F_n$ satisfy the required condition given in Theorem 2.2.
For $O^n F_n \in O^n MHC_0$, we note that $|A_k| \le 1$ and $|B_k| \le 1$. Now for convolution function $O^n f_n * O^n F_n$, we obtain

$$\sum_{k=1}^{\infty} k^2 \left[1+(k-1)\gamma\right]^n \left(|a_k||A_k|\right) + \sum_{k=1}^{\infty} k^2 \left[1+(k-1)\gamma\right]^n \left(|b_k||B_k|\right)$$
$$\le \sum_{k=1}^{\infty} k^2 \left[1+(k-1)\gamma\right]^n |a_k| + \sum_{k=1}^{\infty} k^2 \left[1+(k-1)\gamma\right]^n |b_k| \le 1.$$

Therefore $O^n f_n * O^n F_n \in O^n MHC_\circ$, this proves the required result. □

Theorem 4.2. *If* $O^n f_n$ *and* $O^n F_n$ *of the form* (9) *and* (10) *belong to the class* $O^n MHC_0$, *then the geometric convolution* $O^n f_n \bullet O^n F_n$ *also belong to* $O^n MHC_0$.

Proof: Since $O^n f_n, O^n F_n \in O^n MHC_0$, it follows that

$$\sum_{k=1}^{\infty} k^2 \left[1+(k-1)\gamma\right]^n (|a_k| + |b_k|) \le 1$$

and

$$\sum_{k=1}^{\infty} k^2 \left[1+(k-1)\gamma\right]^n (|A_k| + |B_k|) \le 1.$$

Hence by Cauchy-Schwartz's inequality it is noted that

$$\sum_{k=1}^{\infty} k^2 \left[1+(k-1)\curlywedge\right]^n (\sqrt{|a_k A_k|} + \sqrt{|b_k B_k|}) \le 1.$$

The proof is complete. □

Theorem 4.3. *If* $O^n f_n$ *and* $O^n F_n$ *of the form* (8) *and* (9) *belong to the class* $O^n MHC$ *then the integral convolution* $O^n f_n \cdot O^n F_n$ *also belong to the* $O^n MHC_\circ$.

Proof: Since $O^n f_n, O^n F_n \in O^n MHC_0$, it follows that $|A_k| \le 1$ and $|B_k| \le 1$. Then $O^n f_n \cdot O^n F_n \in O^n MHC_0$ because

$$\sum_{k=1}^{\infty} k^2 \left[1+(k-1)\lambda\right]^n \frac{|a_k A_k|}{k} + \sum_{k=1}^{\infty} k^2 \left[1+(k-1)\lambda\right]^n \frac{|b_k B_k|}{k}$$

$$\le \sum_{k=1}^{\infty} k^2 \left[1+(k-1)\lambda\right]^n \frac{|a_k|}{k} + \sum_{k=1}^{\infty} k^2 \left[1+(k-1)\lambda\right]^n \frac{|b_k|}{k}$$

$$\le \sum_{k=1}^{\infty} k^2 \left[1+(k-1)\lambda\right]^n |a_k| + \sum_{k=1}^{\infty} k^2 \left[1+(k-1)\lambda\right]^n |b_k| \le 1.$$

This proves the required result. □

5. Convex Combinations

In this section, we show that the class $O^n MHC_\circ$ is invariant under convex combinations of its members.

Theorem 5.1. *The class $O^n MHC_0$ is closed under convex combinations.*

Proof: For $i = 1,2,3,...$ suppose that $O^n f_i(z) \in O^n MHC_0$, where $O^n f_i$ is given by

$$O^n f_i(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} \left[1+(k-1)\gamma\right]^n a_{ik} z^k + \sum_{k=1}^{\infty} \left[1+(k-1)\gamma\right]^n \overline{b_{ik} z^k},$$

$a_{ik} \ge 0, b_{ik} \ge 0,$ then by Theorem 2.2,

$$\sum_{k=1}^{\infty} k^2 [1+(k-1)\gamma]^n (|a_i k| + |b_i k|) \le 1.$$

For $\sum_{k=1}^{\infty} t_i = 1, 0 \le t_i \le 1,$ the convex combinations of $O^n f_i$ may be written as

$$\sum_{i=1}^{\infty} t_i O^n f_i(z) = (-1)^n / z + \sum_{k=1}^{\infty} [1+(k-1)\gamma]^n (\sum_{i=1}^{\infty} t_i a_{ik}) z^k + \sum_{k=1}^{\infty} [1+(k-1)\gamma]^n \overline{(t_i b_{ik}) z^k}.$$

Then by

$$\sum_{k=1}^{\infty} k^2 \left[1+(k-1)\lambda\right]^n \left(|a_{ik}| + |b_{ik}|\right) \le 1.$$

$$\sum_{k=1}^{\infty} k^2 \left[1+(k-1)\lambda\right]^n \left(\left|\sum_{i=1}^{\infty} t_i a_{ik}\right| + \left|\sum_{i=1}^{\infty} t_i b_{ik}\right|\right)$$

$$= \sum_{i=1}^{\infty} t_i \left[\sum_{k=1}^{\infty} k^2 \left[1+(k-1)\lambda\right]^n \left(|a_{ik}| + |b_{ik}|\right)\right]$$

$$\le \sum_{i=1}^{\infty} t_i = 1.$$

Thus $\sum_{i=1}^{\infty} t_i O^n f_i(z) \in O^n MHC_0.$ □

Acknowledgements

The work here is supported by MOHE grant FRGS/1/2016/STG06/UKM/01/1.

References

Avkhadiev F.G. & Wirths K.J. 2005. Concave schlict functions with bounded opening angle at infinity. *Lobachevskii Journal of Mathematics* **17**:3–10.

Chuaqui M., Duren P. & Osgood B. 2012. Concave conformal mappings and pre-vertices of Schwarz-Christoffel mappings. *Proceedings of the American Mathematical Society* **140**: 3495–3505.

Clunie J. & Sheil-Small T. 1984. Harmonic univalent functions. Annales Academiae Scientiarum Fennicae. *Series A.I.Mathematica* **9**: 3–25.

Hengartner W. & Schober G. 1987. Univalent harmonic functions. *Transactions of the American Mathematical Society* **299**: 1–31.

AL-Shaqsi K. & Darus M. 2008. On harmonic uinvalent functions with respect to k-symmetric points. *International Journal of Contemporary Mathematical Sciences* **3**: 111-118.

Jahangiri J.M. & Silverman H. 1999. Meromorphic univalent harmonic functions with negative coefficients. *Bulletin-Korean Mathematical Society* **36**(4): 763-770.

Jahangiri J. M. 2000. Harmonic meromorphic starlike functions. *Bulletin-Korean Mathematical Society* **37**(2): 291-302.

Jahangiri J. M. 1998. Coefficient bounds and univalence criteria for harmonic functions with negative coefficients. *Annales Universitatis Mariae Curie-Skłodowska A* **52**(2): 57-66.

Aldawish I. & Darus M. 2015. On Certain Class of Meromorphic Harmonic Concave Functions. *Tamkang Journal of Mathematics* **46**(2): 101-109.

Challab K. & Darus M. 2015. On certain class of meromorphic harmonic concave functions defined by Salagean operator. *Journal of Quality Measurement and Analysis* **11**(1):49-60.

Al-Oboudi F. M. 2004. On univalent functions defined by a generalized Sălăgean operator. *International Journal of Mathematics and Mathematical Sciences* **27**: 1429–1436.

Aldawish I. & Darus M. 2015. On *p*-valent concave functions based on Salagean differential operator. *Kochi J. Math.* **10**: 43-51.

Darus M., Aldawish I. & Ibrahim R. 2015. Some Concavity Properties for general integral operators. *Bulletin of the Iranian Math. Soc.* **41**(5): 1085-1092.

Aldawish I., Darus M. & Ibrahim R.W. 2014. Concavity of Ruscheweyh differential operator. *Acta Universitatis Apulensis* **37**:73-82.

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi
Selangor DE, MALAYSIA
E-mail: khalid_math1363@yahoo.com, maslina@ukm.edu.my[*]

*Corresponding author