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GENERALISED CLASS OF STARLIKE FUNCTIONS OF KOEBE TYPE WITH COMPLEX ORDER

(Kelas Fungsi Bak Bintang Teritlak Jenis Koebe Peringkat Kompleks)

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ABSTRACT

In this paper, we investigate some results on subordination, superordination, best dominant result and the sandwich theorem of the new class $\mathcal{H}_b(\alpha,\beta,\eta)$ of analytic functions with Koebe type. Further, by making use of Jack's Lemma as well as several differential and other inequalities, sufficient condition for starlikeness of the class $\mathcal{H}_b(\alpha,\beta,\eta)$ of *n*-fold symmetric analytic functions of Koebe type is derived. Relevant connections of the results presented here with those given in earlier works are also indicated.

Keywords: subordination; superordination; analytic function; starlikeness; Koebe type

ABSTRAK

Dalam makalah ini dikaji beberapa hasil berkenaan dengan subordinasi, superordinasi, dominan terbaik dan teorem himpitan bagi kelas fungsi baharu $\mathcal{H}_b(\alpha,\beta,\eta)$ yang merupakan fungsi analisis jenis Koebe. Selanjutnya, dengan menggunakan Lema Jack bersama beberapa ketaksamaan pembeza dan ketaksamaan yang lain, syarat cukup untuk kebakbintangan bagi kelas fungsi $\mathcal{H}_b(\alpha,\beta,\eta)$ yang analisis jenis Koebe simetri lipat-*n* diperoleh. Hubungan yang berkaitan dengan kajian-kajian terdahulu juga dinyatakan.

Kata kunci: subordinasi; superordinasi; fungsi analisis; kebakbintangan; jenis Koebe

1. Introduction

Let $H(U)$ denote the class of analytic functions in the open unit disc $U=\{z:|z|<1\}$ of complex plane C. Let $H[a,1]$ denote the subclass of the functions $f\in H(U)$ of the form $f(z)=a+a_1z+a_2z^2+\ldots,\ a\in C$. Also, let $\mathcal{A}$ be the subclass of the function $f\in H(U)$ of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k,\ (z\in U). \tag{1}$$

If the function f and g are analytic in U, we say that the function f is subordinate to g or g is superordinate to f, written as $f\prec g$ or $f(z)\prec g(z)$, if there exist Schwarz function $w(z)$ such that $f(z)=g\big(w(z)\big)$ with $w(0)=0$ and $|w(z)<1|$ in U.

Suppose that h and k are two analytic functions in U, let $\varphi: C^3\times U\to C$. If h and $\varphi\big(h(z),zh'(z),z^2h''(z);z\big)$ are univalent functions in U and if h satisfies the second order superordination

$$k(z)\prec\varphi\big(h(z),zh'(z),z^2h''(z);z\big), \tag{2}$$

then k is said to be a solution of the differential superordination (2). A function $q \in H(U)$ is called a subordinant to (2), if $q(z) \prec h(z)$ for all the functions h satisfying (2).

A univalent subordination $\tilde{q}$ that satisfies $q(z) \prec \tilde{q}(z)$ for all of the subordinants q of (2), is said to be the best subordinant. Miller and Mocanu (2003) obtained the sufficient conditions of the functions k, q and φ for which the following implications hold

$$k(z) \prec \varphi\left(h(z), zh'(z), z^2 h''(z); z\right) \Rightarrow q(z) \prec h(z).$$

Let S^* and $\tilde{S}^*(\alpha)$ be the familiar classes of starlike functions in U and strongly starlike functions of order α in U $(0 < \alpha \le 1)$, if it satisfies

$$S^* = \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ (z \in U) \right\} \tag{3}$$

and

$$\tilde{S}^*(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \alpha\frac{\pi}{2},\ z \in U, 0 < \alpha \le 1 \right\}, \tag{4}$$

respectively. Also $\tilde{S}^*(\alpha) \subset S^*$ and $\tilde{S}^*(1) = S^*$.

Next, we let $\mathcal{H}(\alpha,\beta,\eta)$ the class of functions $f \in A$, for $\alpha > 0, |\beta + \eta i| \le 1$, β and $\eta \in R$ of real numbers, that is

$$\operatorname{Re}\left\{ \left(\frac{zf'(z)}{f(z)}\right)^{\beta+\eta i} \left(\frac{zf'(z)}{f(z)} + \alpha z^2 \frac{zf''(z)}{f'(z)}\right) \right\} > 0,\ z \in U. \tag{5}$$

The class $\mathcal{H}(\alpha) = \mathcal{H}(\alpha,0,0)$ was first introduced by Mocanu (1969), which was then known as the class α convex (or α-starlike) functions. Later, Miller *et al.* (1973) showed that $\mathcal{H}(\alpha)$ is subclass of S^* for any real number α and also that $\mathcal{H}(\alpha)$ is a subclass of convex function K, for $\alpha \ge 1$. Note that $\mathcal{H}(0) = S^*$ and $\mathcal{H}(1) = K$.

Kamali and Srivastava (2004) derive the sufficient conditions for starlikeness of n-fold symmetric function f_b of Koebe type, which is defined by

$$f_b(z) := \frac{z}{\left(1 - z^n\right)^b} \qquad \left(b \ge 0; n \in N := \{1,2,3,\ldots\}\right), \tag{6}$$

which obviously corresponds to the familiar Koebe function when $n = 1$ and $b = 2$. From (5) and (6), the class of function $\mathcal{A}$ with complex order is defined by

$$\mathcal{H}_b(\alpha,\beta,\eta), = \operatorname{Re}\left\{ \left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\eta i} \left(\frac{zf_b'(z)}{f_b(z)} + \alpha z^2 \frac{zf_b''(z)}{f_b'(z)}\right) \right\} > 0,\ z \in U. \tag{7}$$

We note again that $\mathcal{H}(0,0,0) = \mathcal{H}(0) \subset S^*(0)$. $\mathcal{H}(1,0,0) = \mathcal{H}(1) \subset \tilde{S}^*(1/2)$ was investigated by Ramesha *et al.* (1995), $\mathcal{H}(1,0,0) = \mathcal{H}(1) \subset \tilde{S}^*(\gamma)$ where $(\gamma < 1/2)$ already observed by Nunokawa *et al.* (1996), $\mathcal{H}(\alpha,0,0) = \mathcal{H}(\alpha) \subset S^*$ was discussed by Kamali and Srivastava (2004) and $\mathcal{H}(\alpha,\beta,0) = \mathcal{H}(\alpha,\beta) \subset S^*$, was studied by Siregar (2011).

In this paper, we investigate the subordination, superordination, best dominant, best subordinant, sandwich theorem and sufficient conditions for starlikeness of *n*-fold symmetric function f_b of Koebe type and its application of the class denote by $\mathcal{SK}_b(\alpha,\beta,\eta)$. The work of Bansal and Raina (2010), Kamali and Srivastava (2004), and Siregar (2011) have motivated us to come to these problems. See also Shanmugam *et al.* (2007) and Siregar *et al.* (2010) for different studies.

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known results.

Lemma 2.1. (Miller & Mocanu 1985) *Let the function* $q(z)$ *be univalent in the open unit disc* U *and let the function* θ *and* ϕ *be analytic in a domain* D *containing* $q(U)$, *with* $\phi(w)\neq 0$ *when* $w\in q(U)$. *Set*

$$Q(z)=zq'(z)\phi(q(z)) \text{ and } h(z)=\theta(q(z))+Q(z). \tag{8}$$

Suppose that

(i) $Q(z)$ *is starlike univalent* U *and*

(ii) $\operatorname{Re}\left\{\dfrac{zh'(z)}{Q(z)}\right\}>0$ *for* $z\in U$.

If $\theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(q(z))$ *then* $p(z)\prec q(z)$ *and* q *is the best dominant of the subordination.*

Lemma 2.2. (Bulbuača 2002) *Let* $q(z)$ *be univalent in the unit disk* U *and let* ϑ *and* φ *be analytic in a domain* D *containing* $q(U)$. *Suppose that,*

(i) $zq'(z)\varphi(q(z))$ *is univalent and starlike in* U

(ii) $\operatorname{Re}\left\{\dfrac{\vartheta'(q(z))}{\varphi(q(z))}\right\}>0$ *for* $z\in U$.

If $p(z)\in H[q(0),1]\cap Q$ *with* $p(U)\subseteq D$ *and* $\vartheta p(z)+zp'(z)\varphi(p(z))$ *is univalent in* U *and* $\vartheta(q(z))+zq'(z)\varphi(q(z))\prec\vartheta(p(z))+zp'(z)\varphi(p(z))$ *then* $q(z)\prec p(z)$ *and* $q(z)$ *is the best subordinant.*

Lemma 2.3. (Jack 1971) L*et the (nonconstant) function* $w(z)$ *be analytic in the unit disk* U *and such that* $w(0)=0$. *If* $|w(z)|$ *attains its maximum value on circle* $|z|=r=1$ *at a point* $z_0\in U$, *we have* $z_0w'(z)=kw(z_0)$, *where* $k\geq 1$ *is a real number.*

Lemma 2.4. (Fukui *et al.* 1992) *The function* f_b *defined by* (6) *is univalent if and only if*

$$0\leq nb\leq 2. \tag{9}$$

Furthermore, the condition in (9) *is necessary and sufficient for* f_b *to be a starlike function.*

Lemma 2.5. (Miller & Mocanu 1978) *Let* $\Theta(u,v)$ *be a complex-valued function such that* $\Theta: D \to C$ $(D \subset C \times C)$. Here C *being (as usual) the complex plane. Let* $u = u_1 + iu_2$ *and* $v = v_1 + iv_2$. *Supposed that the function* $\Theta(u,v)$ *satisfy each of the following conditions:*

(i) $\Theta(u,v)$ *is continuous in* D,

(ii) $(1,0) \in D$ *and* $\mathrm{Re}(\Theta(1,0)) > 0$,

(iii) $\mathrm{Re}(\Theta(iu_2, v_1)) \le 0$ *for all* $(iu_2, v_1) \in D$ *such that* $v_1 \le -\frac{1}{2}(1+u_2^2)$.

Let $p(z) = 1 + p_1 z + p_2 z + \cdots$ *be analytic (regular) in* U *such that* $(p(z), zp'(z)) \in D$, $(z \in U)$. *If* $\mathrm{Re}(\Theta(p(z), zp'(z))) \in D$, $(z \in U)$, *then* $\mathrm{Re}(p(z)) > 0$ $(z \in U)$.

3. Subordination and Superordination Results

The method of proving in the next theorem is similar to Bansal and Raina (2010), and Siregar (2011).

Theorem 3.1. *Let* $f(z) \in A$ *satisfy* $f(z) \ne 0, (z \in U)$. *Also let the function* $q(z)$ *be univalent in* U, *with* $q(0) = 1$ *and* $q(z) \ne 0$, *such that for* β and $\eta \in R$,

$$\mathrm{Re}\left\{1 + (\beta + \eta i) z \frac{q'(z)}{q(z)} + z \frac{q''(z)}{q'(z)}\right\} > 0, \ (z \in U)$$

and

$$\mathrm{Re}\left\{(\beta + \eta i + 2) q(z) + \frac{1}{\alpha}(\beta + \eta i + 1) + (\beta + \eta i)\left(\frac{zq'(z)}{q(z)} - 1\right) + z\frac{q''(z)}{q'(z)}\right\} > 0, \ (z \in U) \tag{10}$$

for $|\beta + \eta i| \le 1$ *and* $\alpha > 0$. *If*

$$\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\eta i}\left(\frac{zf_b'(z)}{f_b(z)} + \alpha z^2 \frac{f_b''(z)}{f_b''(z)}\right) \prec h(z), \ (z \in U), \tag{11}$$

then $\left(\frac{zf_b'(z)}{f_b(z)}\right) \prec q(z)$ $(z \in U)$ *and* $q(z)$ *is the best dominant of* (11), *where*

$$h(z) = \alpha[q(z)]^{\beta+\eta i+2} + (1-\alpha)[q(z)]^{\beta+\eta i+1} + \alpha z q'(z)[q(z)]^{\beta+\eta i}. \tag{12}$$

Proof. Set p be defined by

$$p(z) = \frac{zf_b'(z)}{f_b(z)}. \tag{13}$$

and

$$\theta(w) = w^{\beta+\eta i}((1-\alpha)w + \alpha w^2) \ \text{ and } \ \phi(w) = w^{\beta+\eta i}.$$

Then $\theta(w)$ and $\phi(w)$ are analytic inside the domain D^* which contains $q(U)$, $q(0) = 1$ and $\phi(w) \ne 0$ when $w \in q(U)$.

Also, we let

$$Q(z)=\alpha zq'(z)\phi((q(z))=\alpha zq'(z)[q(z)]^{\beta+\eta i},$$

and

$$h(z)=\theta\left[q(z)\right]+Q(z)=\alpha[q(z)]^{\beta+\eta i+2}+(1-\alpha)[q(z)]^{\beta+\eta i+1}+\alpha zq'(z)[q(z)]^{\beta+\eta i},$$

then it follows from (10) and (11) that $Q(z)$ is starlike in U and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}=\operatorname{Re}\left\{(\beta+\eta i+2)q(z)+\frac{1}{\alpha}(\beta+\eta i+1)+(\beta+\eta i)\left(\frac{zq'(z)}{q(z)}-1\right)+z\frac{q''(z)}{q'(z)}\right\}>0.$$

We also note that the function $p(z)$ is analytic in U, with $p(0)=q(0)=1$. Since $0\notin p(U)$, therefore $p(U)\subset D^*$. Hence, the hypothesis of Lemma 2.1 are satisfied.

Since $p(z)=\dfrac{zf_b'(z)}{f_b(z)}$, then we have $p'(z)=\dfrac{zf_b''(z)}{f_b(z)}+\dfrac{f_b'(z)}{f_b(z)}-z\left[\dfrac{f_b'(z)}{f_b(z)}\right]^2$.

Multiplying $p'(z)$ with z, we have

$$zp'(z)=\frac{z^2f_b''(z)}{f_b(z)}+\frac{zf_b'(z)}{f_b(z)}-\left[\frac{zf_b'(z)}{f_b(z)}\right]^2.$$

Therefore

$$\frac{z^2f_b''(z)}{f_b(z)}=zp'(z)-\frac{zf_b'(z)}{f_b(z)}+\left[\frac{zf_b'(z)}{f_b(z)}\right]^2=zp'(z)-p(z)+\left[p(z)\right]^2.$$

Applying Lemma 2.1, we find that

$$\begin{aligned}
&\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\eta i}\left(\frac{zf_b'(z)}{f_b(z)}+\alpha z^2\frac{f_b''(z)}{f_b'(z)}\right)\\
&=\left[p(z)\right]^{\beta+\eta i}\left[p(z)+\alpha\left(zp'(z)-p(z)+\left[p(z)\right]^2\right)\right]\\
&=\alpha\left[p(z)\right]^{\beta+\eta i+2}+(1-\alpha)\left[p(z)\right]^{\beta+\eta i+1}+\alpha zp'(z)\left[p(z)\right]^{\beta+\eta i}\\
&\qquad=\theta\left(p(z)\right)+\alpha zp'(z)[p(z)]^{\beta+\eta i}\prec h(z)\\
&=\alpha\left[q(z)\right]^{\beta+\eta i+2}+(1-\alpha)\left[q(z)\right]^{\beta+\eta i+1}+\alpha zq'(z)\left[p(z)\right]^{\beta+\eta i}\\
&\qquad=\theta\left(q(z)\right)+\alpha zq'(z)[q(z)]^{\beta+\eta i}\quad (z\in U).
\end{aligned}$$

Thus in view of Lemma 2.1, we get

$\left(\dfrac{zf_b'(z)}{f_b(z)}\right)\prec q(z)\quad (z\in U)$ and $q(z)$ is the best dominant of (11). □

Theorem 3.2. *Let f be analytic in* U *such that* $f(0)=0$, h *be convex univalent in* U *and* $h\in H[0,1]\cap\mathbb{Q}$. *Assume that*

$$\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\eta i}\left(\frac{zf_b'(z)}{f_b(z)}+\alpha z^2\frac{f_b''(z)}{f_b'(z)}\right)$$

is a univalent function in U, *where* $|\beta+\eta i|\le 1$ *and* $\alpha>0$. *If* $h\in A$ *and the subordination*

$$h(z) = \theta(q(z)) + \alpha z q'(z)[q(z)]^{\beta+\eta i} \prec \left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\eta i}\left(\frac{zf_b'(z)}{f_b(z)} + \alpha z^2 \frac{f_b''(z)}{f_b'(z)}\right)$$

holds then $q(z) \prec \left(zf_b'(z)/f_b(z)\right)$ *implies that* $q(z) \prec p(z)$, *where* $p(z) = \left(zf_b'(z)/f_b(z)\right)$ *and* $q(z)$ *is the best subordinant.*

Proof. Our aim is to apply Lemma 2.2. By setting

$$p(z) = \frac{zf_b'(z)}{f_b(z)}, \quad \vartheta(w) = w^{\beta+\eta i}((1-\alpha)w + \alpha w^2) \text{ and } \quad \varphi(w) = w^{\beta+\eta i} \tag{14}$$

then $\vartheta(w)$ and $\varphi(w)$ are analytic inside the domain D^* which contains $p(U)$, $p(0) = 1$ and $\varphi(w) \neq 0$ when $w \in p(U)$.

It can be observed that $\vartheta(w)$, $\varphi(w)$ are analytic in *U*. Thus,

$$\operatorname{Re}\left\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\right\} > 0.$$

Now, we show that

$$h(z) = \vartheta(q(z)) + \alpha z q'(z)\varphi(q(z)) \prec \varphi(p(z)) + \alpha z p'(z)\varphi(p(z)).$$

By the assumption of the theorem

$$\begin{aligned} h(z) &= \vartheta(q(z)) + \alpha z q'(z)[q(z)]^{\beta+\eta i} \\ &= \alpha[q(z)]^{\beta+\eta i+2} + (1-\alpha)[q(z)]^{\beta+\eta i+1} + \alpha z q'(z)[q(z)]^{\beta+\eta i} \\ &\prec \alpha[p(z)]^{\beta+\eta i+2} + (1-\alpha)[p(z)]^{\beta+\eta i+1} + \alpha z p'(z)[p(z)]^{\beta+\eta i} \\ &= \vartheta(p(z)) + \alpha z p'(z)[p(z)]^{\beta+\eta i} \\ &= \left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\eta i}\left(\frac{zf_b'(z)}{f_b(z)} + \alpha z^2 \frac{f_b''(z)}{f_b'(z)}\right). \end{aligned}$$

Thus in view of Lemma 2.2, $q(z) \prec p(z)$ which implies $q(z) \prec zf_b'(z)/f_b(z)$ and q is the best subordinant. □

If we combine Theorem 3.1 and Theorem 3.2, then we obtain the differential *Sandwich-Type* theorem. For $\beta + \eta i = 1$ and $\alpha = 1$ in Theorem 3.1, then we get the following interesting corollary.

Corrollary 3.1. *Let* $f(z) \in A$ *satisfy* $f(z) \neq 0$, $(z \in U)$. *Also let the function q(z) be univalent in U with* $q(0) = 1$ *and* $q(z) \neq 0$, *such that*

$$\operatorname{Re}\left\{1 + q(z) - \frac{zq'(z)}{q(z)} + z\frac{q''(z)}{q'(z)}\right\} > 0, \ (z \in U) \tag{15}$$

If

$$\left(1+\frac{zf_b''(z)}{f_b'(z)}\right) \prec q(z)-\frac{zq'(z)}{q(z)},\ (z\in U), \tag{16}$$

then $zf_b'(z)/f_b(z) \prec q(z),\ (z\in U)$.

4. The properties of the class $\mathcal{SK}_b(\alpha,\beta,\eta)$

We begin by proving a stronger result than what we indicated in the preceding section. The method of proving is similar to Kamali and Srivastava (2004).

Theorem 4.1. *Let the n-fold symmetric function* $f_b(z)$ *defined by* (6), *be analytic in* U, *with* $f_b(z)/z \neq 0\ (z\in U)$.

(i) *If* $f_b(z)$ *satisfies the inequality:*

$$\operatorname{Re}\left\{\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\eta i}\left[\frac{zf_b'(z)}{f_b(z)}+\alpha\frac{z^2 f_b''(z)}{f_b(z)}\right]\right\}$$

$$>\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right]. \tag{17}$$

Then, $f_b(z)$ *is starlike in* U *for* $\alpha>0$ *and* $|\beta+\eta i|\leq 1$ *and*

$$0\leq nb<2; \frac{3\alpha+2-\sqrt{\Delta^*}}{2\alpha}\leq nb\leq\frac{3\alpha+2+\sqrt{\Delta^*}}{2\alpha};\ (\Delta^*:=9\alpha^2-4\alpha+4).$$

(ii) *If* $f_b(z)$ *satisfies the inequality* (17) *with* $\alpha=0$ *and* $\beta+\eta i=0$ *that is, if*

$$\operatorname{Re}\left(\frac{zf_b'(z)}{f_b(z)}\right)>1-\frac{nb}{2},\ (z\in U). \tag{18}$$

Then, $f_b(z)$ *is starlike in* U *for* $0\leq nb<2$.

(iii) *If* $f_b(z)$ *satisfies the inequality* (17) *with* $\beta+\eta i=1$, *that is, if*

$$\operatorname{Re}\left(\alpha\frac{z^2 f_b''(z)}{f_b(z)}+\frac{zf_b'(z)}{f_b(z)}\right)>-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right),\ (z\in U). \tag{19}$$

Then, $f_b(z)$ *is starlike in* U *for*

$$\alpha>0 \ \text{ and } \ \frac{3\alpha+2-\sqrt{\Delta^*}}{2\alpha}\leq nb\leq\frac{3\alpha+2+\sqrt{\Delta^*}}{2\alpha};\ (\Delta^*:=9\alpha^2-4\alpha+4).$$

Proof. (i) Let $\alpha>0$ and $f_b(z)$ satisfy the hypothesis of Theorem 4.1. We put

$$\frac{zf_b'(z)}{f_b(z)}=\frac{1+(nb-1)w(z)}{1-w(z)}, \tag{20}$$

where $w(U)$ is analytic in U, with $w(0)=0$ and $w(z)\neq 1$, $(z\in U)$. Then, we have

$$\frac{\left[f_b'(z)+zf_b''(z)\right]f_b(z)-z\left[f_b'(z)\right]^2}{\left[f_b(z)\right]^2}=\frac{(nb-1)\,w'(z)(1-w(z))+w'(z)\left[1+(nb-1)\,w(z)\right]}{(1-w(z))^2},$$

$$=\frac{(nb-1)\,w'(z)(1-w(z))+w'(z)\left[1+(nb-1)\,w(z)\right]}{(1-w(z))^2}$$

$$=\frac{nbw'(z)-w'(z)-nbw(z)w'(z)+w(z)w'(z)+w'(z)+nbw(z)w'(z)-w(z)w'(z)}{(1-w(z))^2}$$

$$=\frac{nbw'(z)}{\left[1-w(z)\right]^2},$$

which implies that

$$\frac{zf_b''(z)}{f_b'(z)}+\frac{f_b'(z)}{f_b(z)}-z\left[\frac{f_b'(z)}{f_b(z)}\right]^2=\frac{nbw'(z)}{\left[1-w(z)\right]^2}. \tag{21}$$

On the other hand, we can write

$$\frac{z^2 f_b''(z)}{f_b(z)}=zp'(z)-\frac{zf_b'(z)}{f_b(z)}+\left[\frac{zf_b'(z)}{f_b(z)}\right]^2$$

$$=\frac{nbw'(z)}{\left[1-w(z)\right]^2}-\frac{1+(nb-1)\,w(z)}{1-w(z)}+\left(\frac{1+(nb-1)\,w(z)}{1-w(z)}\right)^2$$

and

$$\alpha\frac{z^2 f_b''(z)}{f_b(z)}=\alpha\left(\frac{nbw'(z)}{\left[1-w(z)\right]^2}-\frac{1+(nb-1)\,w(z)}{1-w(z)}+\left(\frac{1+(nb-1)\,w(z)}{1-w(z)}\right)^2\right)$$

$$=\alpha\left(\frac{nbw'(z)}{\left[1-w(z)\right]^2}+\left(\frac{1+(nb-1)\,w(z)}{1-w(z)}\right)^2\right)-\alpha\left(\frac{1+(nb-1)\,w(z)}{1-w(z)}\right),$$

which in turn, implies that

$$\left[\frac{zf_b'(z)}{f_b(z)}+\alpha\frac{z^2 f_b''(z)}{f_b(z)}\right]=\left(\frac{1+(nb-1)\,w(z)}{1-w(z)}\right)+$$

$$\left[\alpha\left(\frac{nbw'(z)}{\left[1-w(z)\right]^2}+\left(\frac{1+(nb-1)\,w(z)}{1-w(z)}\right)^2\right)-\alpha\frac{1+(nb-1)\,w(z)}{1-w(z)}\right],$$

$$=\alpha\left(\frac{nbw'(z)}{\left[1-w(z)\right]^2}+\left(\frac{1+(nb-1)\,w(z)}{1-w(z)}\right)^2\right)+(1-\alpha)\frac{1+(nb-1)\,w(z)}{1-w(z)}.$$

Therefore

$$\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\eta i}\left[\frac{zf_b'(z)}{f_b(z)}+\alpha\frac{z^2f_b''(z)}{f_b(z)}\right]$$

$$=\left(\frac{1+(nb-1)w(z)}{1-w(z)}\right)^{\beta+\eta i}\left[\alpha\left(\frac{nbw'(z)}{[1-w(z)]^2}+\left(\frac{1+(nb-1)w(z)}{1-w(z)}\right)^2\right)+(1-\alpha)\frac{1+(nb-1)w(z)}{1-w(z)}\right].$$

Now, we claim that $w(z)<1,\ (z\in U)$. If there exist z_0 a in U such that $|w(z_0)|=1$, then by Lemma 2.3, we have $z_0w'(z)=kw(z_0)$, where $k\geq 1$ is a real number. By setting $w(z_0)=e^{i\theta}\ (0\leq\theta\leq 2\pi)$, thus we find that

$$\operatorname{Re}\left\{\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\eta i}\left[\frac{zf_b'(z)}{f_b(z)}+\alpha\frac{z^2f_b''(z)}{f_b(z)}\right]\right\}$$

$$=\operatorname{Re}\left(\frac{1+(nb-1)w(z_0)}{1-w(z_0)}\right)^{\beta+\eta i}$$

$$\left[\alpha\left(\frac{nbw'(z_0)}{[1-w(z_0)]^2}+\left(\frac{1+(nb-1)w(z_0)}{1-w(z_0)}\right)^2\right)+(1-\alpha)\left(\frac{1+(nb-1)w(z_0)}{1-w(z_0)}\right)\right],$$

$$=\operatorname{Re}\left(\frac{1+(nb-1)e^{i\theta}}{1-e^{i\theta}}\right)^{\beta+\eta i}\left[\alpha\left(\frac{nbke^{i\theta}}{\left[1-e^{i\theta}\right]^2}+\left(\frac{1+(nb-1)e^{i\theta}}{1-e^{i\theta}}\right)^2\right)+(1-\alpha)\left(\frac{1+(nb-1)e^{i\theta}}{1-e^{i\theta}}\right)\right],$$

$$=\operatorname{Re}\left(\frac{1+(nb-1)e^{i\theta}}{1-e^{i\theta}}\right)^{\beta+\eta i}\left[\alpha\left(\frac{nbke^{i\theta}}{\left[1-e^{i\theta}\right]^2}+\left(\frac{1+(nb-1)e^{i\theta}}{1-e^{i\theta}}\right)^2\right)+(1-\alpha)\left(\frac{1+(nb-1)e^{i\theta}}{1-e^{i\theta}}\right)\right].$$

By taking $e^{i\theta}=\cos\theta+i\sin\theta$, we get

$$=\operatorname{Re}\left(1-\frac{nb}{2}\right)^{\beta+\eta i}\left[\alpha\left(\frac{-nbk}{4\sin^2\left(\frac{\theta}{2}\right)}+\left(1-\frac{nb}{2}\right)^2-\frac{-n^2b^2}{4}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\right)+(1-\alpha)\left(1-\frac{nb}{2}\right)\right],$$

$$=\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[\alpha\left(\frac{-nbk}{4\sin^2\left(\frac{\theta}{2}\right)}+\left(1-\frac{nb}{2}\right)^2-\frac{-n^2b^2}{4}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\right)+(1-\alpha)\left(1-\frac{nb}{2}\right)\right],$$

$$=\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[-\frac{\alpha nb}{4}\left(\frac{k+nbk\cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}\right)+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right],$$

$$\leq\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right]\quad(z\in U),$$

since $k\geq 1$.

If we let

$$\mathrm{Re}\left\{\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\eta i}\left[\frac{zf_b'(z)}{f_b(z)}+\alpha\frac{z^2 f_b''(z)}{f_b(z)}\right]\right\}$$
$$\le\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right]=\tau(nb). \tag{22}$$

Then,

$$\tau(nb)\le 0,\ \left(0\le nb<2;\frac{3\alpha+2-\sqrt{\Delta^*}}{2\alpha}\le nb\le\frac{3\alpha+2+\sqrt{\Delta^*}}{2\alpha};\ (\Delta^*:=9\alpha^2-4\alpha+4)\right).$$

Thus we have

$$\mathrm{Re}\left\{\left(\frac{z_0 f_b'(z_0)}{f_b(z_0)}\right)^{\beta+\eta i}\left[\frac{z_0 f_b'(z_0)}{f_b(z_0)}+\alpha\frac{z_0{}^2 f_b''(z_0)}{f_b(z_0)}\right]\right\}\le 0,$$
$$\left(0\le nb<2;\frac{3\alpha+2-\sqrt{\Delta}}{2\alpha}\le nb\le\frac{3\alpha+2+\sqrt{\Delta}}{2\alpha};\ (\Delta^*:=9\alpha^2-4\alpha+4)\right), \tag{23}$$

which is a contradiction to the hypotheses of (17). Therefore, $|w(z)|<1$ for all in U.

Hence f_b is starlike in U, there by proving the assertion (i) of Theorem 4.1. This complete the proof of our theorem. The proof of the assertion (ii) of Theorem 4.1 was given by Fukui *et al.* (1992) and the proof of the assertion (iii) of Theorem 4.1 was given by Kamali and Srivastava (2004) and so we omit the details.

From the result in Theorem 4.1 (i), we can obtain another one for $0\le nb<2$, so that from Lemma 2.4, the function f_b defined by (6) is univalent and starlike function.

5. Application of Differential Inequalities

We apply the following result involving differential inequalities with a view to deriving several further sufficient conditions for starlikeness of the *n*-fold symmetric function f_b defined by (6) by using Lemma 2.5.

Let us now consider the following implication:

$$\mathrm{Re}\left\{\left(z\frac{f_b'(z)}{f_b(z)}\right)^{\beta+\eta i}\left[z\frac{f_b'(z)}{f_b(z)}+\alpha z^2\frac{f_b''(z)}{f_b(z)}\right]\right\}>$$
$$\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right]\Rightarrow\mathrm{Re}\left\{\left(z\frac{f_b'(z)}{f_b(z)}\right)^{\mu}\right\}>0 \tag{24}$$
$$\left(z\in U;\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right]<1;\ \alpha\ge 0;\mu\ge 1\right).$$

If we put $p(z)=\left\{z\dfrac{f_b'(z)}{f_b(z)}\right\}^{\mu}$ then (24) is equivalent to

$$\operatorname{Re}\left\{\frac{\alpha}{\mu}\{p(z)\}^{\frac{1-\mu}{\mu}}zp'(z)+\alpha\{p(z)\}^{\frac{2}{\mu}}+(1-\alpha)\{p(z)\}^{\frac{1}{\mu}}\right.$$
$$\left.+\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right]\right\}>0 \tag{25}$$

$\Rightarrow \operatorname{Re}(p(z))>0, \quad (z\in U).$

By setting $p(z)=u$ and $zp'(z)=v,$ and letting

$$\Theta(u,v)=\frac{\alpha}{\mu}u^{\frac{1-\mu}{\mu}}v+\alpha u^{\frac{2}{\mu}}+(1-\alpha)u^{\frac{1}{\mu}}+$$
$$\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right].$$

For $\alpha\ge 0, |\beta+\eta i|\le 1,\ \mu\ge 1$ and from Lemma 2.5, we have

$$\operatorname{Re}(\Theta(1,0))=1+\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right]>0,$$

since

$$\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right]<1.$$

Thus the condition (i) and (ii) of Lemma 2.5 are satisfied. Moreover, for

$$(iu_2,v_1)\in D \text{ such that } v_1\le -\frac{1}{2}\left(1+u_2^2\right),$$

we obtain

$$\operatorname{Re}(\Theta(iu_2,v_1))=\frac{\alpha}{\mu}|u_2|^{\frac{(1-\mu)}{\mu}}v_1\cos\left(\frac{(1-\mu)\pi}{2\mu}\right)+\alpha|u_2|^{\frac{2}{\mu}}\cos\left(\frac{\pi}{\mu}\right)+(1-\alpha)|u_2|^{\frac{1}{\mu}}\cos\left(\frac{\pi}{2\mu}\right)$$
$$+\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right],$$
$$\le -\frac{\alpha}{2\mu}\left(1+u_2^2\right)|u_2|^{\frac{(1-\mu)}{\mu}}\sin\left(\frac{\pi}{2\mu}\right)+\alpha|u_2|^{\frac{2}{\mu}}\cos\left(\frac{\pi}{\mu}\right)+(1-\alpha)|u_2|^{\frac{1}{\mu}}\cos\left(\frac{\pi}{2\mu}\right)$$
$$+\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln(2-nb)}{2}\right]\left[-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right].$$

Which, upon putting $|u_2|=s\ (s>0),$ yields

$$\operatorname{Re}(\Theta(iu_2,v_1))\le\Phi(s), \tag{26}$$

where

$$\Phi(s)=-\frac{\alpha}{2\mu}\left(1+s^2\right)s^{\frac{(1-\mu)}{\mu}}\sin\left(\frac{\pi}{2\mu}\right)+\alpha s^{\frac{2}{\mu}}\cos\left(\frac{\pi}{\mu}\right)+(1-\alpha)s^{\frac{1}{\mu}}\cos\left(\frac{\pi}{2\mu}\right),$$

$$+\left(1-\frac{nb}{2}\right)^{\beta}\left[\frac{\cos\ln\left(2-nb\right)}{2}\right]\left[-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\right]. \tag{27}$$

If we choose $\mu = 1$ and $nb \to 2$, we obtain the following result.

Theorem 5.1. *If n-fold symmetric function* f_b*, defined by* (6) *and analytic in* U *with*

$$\frac{f_b(z)}{z} \neq 0 \ \left(z \in U\right),$$

satisfies the following inequality:

$$\operatorname{Re}\left\{\left(z\frac{f_b'(z)}{f_b(z)}\right)^{\beta+\eta i}\left[z\frac{f_b'(z)}{f_b(z)}+\alpha z^2\frac{f_b''(z)}{f_b(z)}\right]\right\} > 0, \tag{28}$$

then $f_b \in S^*$ *for any real* $\alpha \geq 0$.

Proof. For $\mu = 1$, $nb \to 2$, we find from (27) that

$$\Phi(s) := -\frac{\alpha}{2}\left(1-3s^2\right) \leq 0 \ \ (s \in R),$$

which implies Theorem 5.1 in view of the remark. □

Next, for $\alpha = 2/3$, $nb = 3 \pm \sqrt{3}$ and $\mu = 2$, we get the next result.

Theorem 5.2. *If n-fold symmetric function* f_b*, defined by* (6) *and analytic in* U *with*

$$\frac{f_b(z)}{z} \neq 0 \ \left(z \in U\right),$$

satisfies the following inequality:

$$\operatorname{Re}\left\{\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\beta+\eta i}\left[\frac{zf_b'(z)}{f_b(z)}+\frac{2}{3}\frac{z^2 f_b''(z)}{f_b(z)}\right]\right\} > 0, \tag{29}$$

then

$$\left|\arg\left(\frac{zf_b'(z)}{f_b(z)}\right)\right| < \frac{\pi}{4} \quad (z \in U)$$

or, equivalently,

$$\mathcal{H}_b\left(\frac{2}{3},\beta,\eta\right) \subset \tilde{S}^*\left(\frac{1}{2}\right).$$

Proof. By setting $\alpha = 2/3$, $nb = 3 \pm \sqrt{3}$ and $\mu = 2$, in (27), we have

$$\Phi(s) := -\frac{\sqrt{2s}\left(1-s\right)^2}{12s} \leq 0 \ \ (s > 0),$$

which leads us to Theorem 5.2 just as in the proof of Theorem 5.2. From the result in Theorem 5.2, we can conclude that $\mathcal{H}_b\left(\frac{2}{3},\beta,\eta\right) = \mathcal{H}_b\left(\frac{2}{3},0,0\right) \subset \tilde{S}^*\left(\frac{1}{2}\right)$. □

Remark 5.1. If, for some choices of parameters α, β, μ and nb from Theorem 5.1 and Theorem 5.2, we find that $\Phi(s) \leq 0, (s > 0)$. Then, we can conclude from (26) and Lemma 2.5 that the corresponding implication (24) holds true.

Remark 5.2. For $\eta = 0$, we will obtain the results in Theorem 5.1 and Theorem 5.2 by Siregar (2011) and for $\beta = 1$ and $\eta = 0$, we will have result assert by Kemali and Srivastava (2004).

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