*Journal of Quality Measurement and Analysis* *JQMA* 13(1), 2017, 25-34
*Jurnal Pengukuran Kualiti dan Analisis*

# FULLY RIGHT PURE GROUP RINGS

(Gelanggang Kumpulan Tulen Kanan Penuh)

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*ABSTRACT*

In this paper, we study the pure right ideal in rings, and then the fully right pure rings. The necessary and sufficient conditions for a ring $R$ to be fully right pure ring are determined. Also, we study some properties of fully right pure rings. Finally we study the fully right pure group rings. The necessary and sufficient conditions on a group $G$ and a ring $A$ for the group ring $A[G]$ to be fully right pure group ring are determined.

*Keywords*: pure ideal; fully pure ring; group rings; fully right pure group rings

*ABSTRAK*

Dalam makalah ini, kami mengkaji unggulan kanan tulen dalam gelanggang, dan seterusnya unggulan tulen kanan penuh. Syarat-syarat cukup dan perlu untuk suatu gelanggang $R$ menjadi gelanggang tulen kanan penuh ditentukan. Kami juga mengkaji sifat-sifat bagi gelanggang tulen kanan penuh. Akhirnya kami mengkaji gelanggang kumpulan tulen kanan penuh. Syarat-syarat cukup dan perlu bagi suatu kumpulan $G$ dan suatu gelanggang $A$ bagi gelanggang kumpulan $A[G]$ untuk menjadi gelanggang kumpulan tulen kanan penuh ditentukan.

*Kata kunci*: unggulan tulen; gelanggang tulen penuh; gelanggang kumpulan; gelanggang kumpulan tulen kanan penuh

## 1. Introduction

Group rings are very interesting algebraic structures, their concept is relatively old. The concept became important after the article of Connel, 1963, which gave highly stimulating to the area as well as the inclusion of chapter on group rings in the book by Lambek, 1966. There are many studies that discuss the necessary and sufficient conditions for a ring $R$ to have all its ideals in the same class, that is, all right ideals are prime (Tsutsui 1996), all right ideals are semi prime (Ahmad 2008), or all right ideals are weakly prime (Hirano *et al.* 2010). The aim in this paper is to determine the necessary and sufficient conditions on a group $G$ and a ring $A$ for the group ring $A[G]$ to be fully right pure group ring (all its right ideals are pure) and to determine the properties of this kind of group rings.

## 2. Preliminaries

**Notation 2.1**: Throughout this paper the following notation will be adopted

| | |
|---|---|
| $R$ | : Unitary associative ring. |
| $l.a.r.i-ring$ | : Left annihilator right ideal ring. |
| $zc-ring$ | : Zero commutative ring. |
| $zi-ring$ | : Zero insertive ring. |
| $U_R$ | : The set of all units of a ring $R$ . |

| | |
|---|---|
| $N_R$ | : The set of all nilpotent elements of $R$ . |
| $Z_R$ | : The set of all zero divisors of a ring $R$ . |
| $SN_R$ | : The set of all simple nilpotent elements of $R = \{x \in R : x^2 = o\}$ . |
| $I_R$ | : The set of all idempotent elements of $R$ . |
| $AC - ring$ | : Almost commutative ring. |
| $Ann_r(A)$ | : The right annihilator of the set $A$ . |
| $Ann_l(A)$ | : The left annihilator of the set $A$ . |
| $A[G]$ | : The group ring of a group G over a ring $A$ . |
| $\omega(G)$ | : The augmentation ideal of $A[G]$. |
| $w_G[H]$ | : The right ideal of $A[G]$ generated by the set $\{h-1 : h \in H\}$ . |
| $\delta$ | : The augmentation map of the group ring. |

**Definition 2.2.**

(i) An element $x \in R$ is said to be a right (left) strongly regular if there is $y \in R$ such that $x = yx^2$ $x = x^2 y$ .

$R$ is said to be right (left) strongly regular ring if every element of $R$ is right (left) strongly regular.

(ii) A ring $R$ is said to be $\pi$-regular ring if for all $x \in R$ there is $n \in \mathbb{N}$ and $y \in R$ such that $x^n = x^n y x^n$ .

(iii) A ring $R$ is said to be semi $\pi$- regular ring if for all $x \in R$ there is $n \in \mathbb{N}$ and $y \in R$ such that $x^n = x^n yx$ or $x^n = xyx^n$ .

**Remark 2.3.** (Kayyaly 2008) It is well known that $R$ is right strongly regular ring if and only if $R$ is left strongly regular, so we have strongly regular ring only.

**Definition 2.4**.

(i) A ring $R$ is said to be almost commutative ring $(AC - ring)$ if for any proper right prime ideal $P$ of $R$ and $a \notin P$ , there is $x \in R$ such that $ax \in C_R - P$ .

(ii) A ring $R$ is said to be a left annihilator right ideal ( $l.a.r.i - ring$ ) iff, $Ann_l(A)$ is a right ideal of $R$ for all subset $A$ of $R$ .

(iii) A ring $R$ is said to be duo−ring iff, all its ideals are two−sided ideals.

(iv) A ring $R$ is said to be zero−insertive ring ( $zi - ring$ ) iff, for all $a, b \in R$ such that $ab = 0$, implies that $arb = 0$ for all $r \in R$ .

(v) A ring $R$ is said to be semi commutative ring iff, $aR = Ra$ , $\forall\, a \in R$ .

(vi) A ring $R$ is said to be zero−commutative ring ( $zi - ring$ ) iff, for all $a, b \in R$ such that $ab = 0$, implies that $ba = 0$.

**Lemma 2.5**. (Sbah 1999) *For any ring $R$ the following hold:*

*(i) If $R$ is a $zc - ring$ , then $R$ is a $zi - ring$ .*

*(ii) $R$ is a $zi - ring$ iff, $R$ is a $l.a.r.i - ring$ .*

*(iii) If $R$ is a $zi - ring$ , then $I_R \subseteq C_R$ .*

**Lemma 2.6.** (Wong 1979) *If $R$ is $AC-ring$, then every prime (semi prime) ideal is a completely prime (semi prime) ideal.*

**Lemma 2.7.** (Ahmad & Hijjawi 2014) *If $R$ is a unitary ring satisfies one of the following conditions, then $I_R \subseteq C_R$.*

*(i) If $SN_R = \{0\}$.*

*(ii) If $R$ is a zero – commutative ring.*

*(iii) If $R$ is a duo – ring.*

*(iv) If $R$ is a semi commutative ring.*

**Remark 2.8.** If $R$ is a regular ring in which $SN_R = \{0\}$, then $R$ is an $AC-ring$.

**Proof**. Let $P$ be a right prime ideal and $a \notin P$, since $R$ is a regular and $a \in R$, then there is $b \in R$ such that $a = aba \notin P$.

$ab$ is idempotent for $(ab)^2 = abab = (aba)b = ab$. Let $ab = e$, so $ab \in C_R$ since $I_R \subseteq C_R$ (by Lemma 2.7) and $ab \notin P$ for if $ab \in P$, then $aba \in P$ contradiction. Thus $\exists$ $b \in R$ such that $ab \in C_R - P$, therefore $R$ is an $AC-ring$. □

**Lemma 2.9.** *Let $R$ be an $AC-ring$. If $R$ is a fully semi prime right ring, then $R$ is a strongly regular ring.*

**Proof:** Let $a \in R$ and suppose $a \notin a^2R = \bigcap_{i \in \triangle} P_i$ ; $P_i$ is prime ideal (since $a^2R$ is a semi prime right ideal), $\forall i \in \triangle$. Then there is $P_i$ such that $a \notin P_i$ since $R$ is $AC-ring$, then there is $x \in R$ such that $ax \in C_R$ and $ax \notin P_i$. So we have $axRax = axaxR = a(ax)xR = a^2x^2R \subseteq a^2R$. This implies $ax \in a^2R \subseteq P_i$ (since $a^2R$ is a semi prime right ideal) contradiction. So $a \in a^2R$. Therefore there is $r \in R$ such that $a = a^2r$, thus $R$ is strongly regular ring.

**Definition 2.10.** A proper right ideal $I$ of a ring $R$ is said to be pure right ideal if for all $a \in I$, there is $b \in I$ such that $a = ab$. □

**Theorem 2.11.** (Al-Jaleel 1987) *Let $I$ be an ideal of a commutative ring $R$, then $I$ is a pure ideal if and only if for every finite subset $\{a_1, a_2, ..., a_n\}$ of $I$, there exists $b \in I$ such that $a_i b = a_i$ for all $i = 1, 2, ..., n$.*

**Lemma 2.12.** *Let $A$ be a right ideal of a ring $R$. If $I$ is a pure right ideal of $A$, then $I$ is a pure right ideal of $R$.*

**Proof:** Let $M = I + IR$, then it is clear that $M$ is a right ideal of $R$ and we remark.

$I \subseteq I + IR = I + I.IR \subseteq I + I.AR \subseteq I + IA \subseteq I + I = I$ ($I = I.I$ for $I$ is pure right ideal in $A$). So $I = I + IR = M$

This means that, $I$ is a right ideal of $R$ and it is pure right ideal. □

**Theorem 2.13.** *Let $f : R \to S$ be a ring epimorphism. If $I$ is a pure right ideal of $R$, then $f(I)$ is a pure right ideal of $S$.*

**Proof:** Let $f : R \to S$ be a ring epimorphism. Let $I$ be a pure right ideal of $R$ and $y \in f(I)$, then there is $x \in I$ such that $f(x) = y$. Since $x \in I$ and $I$ is pure, then there is $\bar{x} \in I$ such that $x\bar{x} = x$.

Consequently, $f(x).f(\bar{x}) = f(x)$. So $y.\ f(\bar{x}) = y$. But $f(\bar{x}) \in f(I)$, so there is $\bar{y} = f(\bar{x}) \in f(I)$ such that $y\bar{y} = y$. Hence $f(I)$ is a pure right ideal of $S$. □

## 3. Fully Right Pure Ring

**Definition 3.1.** A ring $R$ is said to be a fully right pure ring iff, every proper right ideal $I$ of $R$ is a pure right ideal.

- $R$ is fully left pure ring iff, every proper left ideal of $R$ is a pure left ideal.
- $R$ is fully pure ring if it is fully right and fully left pure ring.

**Theorem 3.2.** (Kayyaly 2008) *Let $R$ be any ring, then the following are equivalent:*

*(i)* $R$ *is a fully right pure ring.*

*(ii)* $R$ *is a strongly regular ring.*

*(iii)* $R$ *is an abelian regular ring.*

**Lemma 3.3.** *Let $R$ be a ring in which $SN_R = \{0\}$, then for all $x \in R$ we have $Ann_r(x)$ is a two sided ideal.*

**Proof:** Let $x \in R$ and $y \in Ann_r(x)$, then $xy = 0$, so $(yx)^2 = yxyx = y(xy)x = 0$, so $yx \in SN_R = \{0\}$. This implies $yx = 0$. Thus $y \in Ann_l(x)$. Therefore $Ann_r(x) \subseteq Ann_l(x)$. In the same way $Ann_l(x) \subseteq Ann_r(x)$, hence $Ann_r(x)$ is a two sided ideal. □

**Lemma 3.4.** (Wong 1972) *Let $R$ be a ring in which $SN_R = \{0\}$ and every completely prime ideal is a maximal right ideal. Then $R = Z_R \cup U_R$.*

**Theorem 3.5.** *The following are equivalent for any ring $R$:*

*(i)* $R$ *is a fully right pure ring.*

*(ii)* $R$ *is a regular $AC-ring$.*

*(iii)* $R$ *is a fully right idempotent $AC-ring$.*

*(iv)* $R$ *is a fully right semi prime $AC-ring$.*

*(v)* $SN_R = \{0\}$ *and every proper completely prime ideal is maximal right ideal.*

*(vi)* $SN_R = \{0\}$ *and $R$ is a semi $\pi-regular$ ring.*

**Proof:** $i \Rightarrow ii$ Since $R$ is a fully right pure ring, then it is strongly regular (Theorem 3.2). Now, let $a \in SN_R$, then $a^2 = 0$ but $R$ is strongly regular, so there is $r \in R$ such that $a = a^2 r = 0.r = 0$, therefore $SN_R = \{0\}$. Since $R$ is strongly regular, then it is regular. We have $R$ is regular and $SN_R = \{0\}$, thus $R$ is $AC-ring$ (Remark 2.8).

$ii \Rightarrow iii$ Clear.

$iii \Rightarrow iv$ Let $I, P$ are two right ideals of $R$ such that $I^2 \subseteq P$. Since $I$ is idempotent right ideal, then $I = I^2 \subseteq P$ this implies $I \subseteq P$, thus $P$ is semi prime right ideal and $R$ is fully right semi prime ring.

$iv \Rightarrow v$ Since $R$ is a fully right semi prime $AC-ring$, then $R$ is a regular ring (Lemma 2.9). Now Let $P$ be any completely prime ideal, and suppose $P \underset{\neq}{\subset} I \subseteq R$ where $I$ is a right ideal of $R$. Then there is $a \in I$ such that $a \notin P$. Since $R$ is a regular ring, there is $b \in R$ such that $a = a\,b\,a$ which implies $a\,b - 1\; a = 0 \in P$, since $P$ is a completely prime ideal and $a \notin P$ we have $a\,b - 1 \in P \subseteq I$, but $a \in I$ implies $a\,b \in I$. Therefore $1 \in I$ which implies $I = R$. This means that $P$ is a maximal right ideal, and we have $SN_R = \{0\}$ (Lemma 2.9).

$v \Rightarrow vi$ Since $SN_R = \{0\}$, then for all $0 \neq x \in R$ we have $Ann_r(x)$ is a two sided ideal (Lemma 3.3), so $\bar{R} = R / Ann_r\; x$ is again a ring in which $SN_{\bar{R}} = \{0\}$ for :

If $\bar{y} \in SN_{\bar{R}} \Rightarrow \bar{y}^2 = \bar{0} \Rightarrow \overline{y^2} = \bar{0} \Rightarrow y^2 \in Ann_r(x) \Rightarrow y^2 x = 0 \Rightarrow y \cdot\; yx \;= 0 \Rightarrow y \in Ann_l(yx) = Ann_r(yx) \Rightarrow yxy = 0 \Rightarrow yxyx = 0 \cdot x = 0 \Rightarrow yx^{\;2} = 0 \Rightarrow yx \in SN_R = \{0\} \Rightarrow yx = 0 \Rightarrow y \in Ann_l(x) = Ann_r(x) \Rightarrow \bar{y} = 0$.

We have $\bar{x} = x + Ann_r\; x$ is not zero divisor for if $\bar{y} \in \bar{R}$ such that $\bar{x}.\bar{y} = \bar{0}$. Then $xy \in Ann_r(x)$ so $xyx = 0$ which implies $xyxy = 0.y = 0$, i.e. $xy^{\;2} = 0$ which implies $xy \in SN_R = \{0\}$ i.e., $xy = 0$ which implies $y \in Ann_r(x)$, so $\bar{y} = \bar{0}$, then (Lemma 3.4) there is $\bar{r}$ such that $\bar{x} \cdot \bar{r} = \bar{1}$ which implies $x \cdot r - 1 \in Ann_r\; x$, so there is $p \in Ann_r(x)$ such that $xr - p = 1$ implies, $xrx^n - px^n = x^n$ which implies $xrx^n - \; px \;\; x^{n-1} = x^n$ implies $xrx^n - 0.\,x^{n-1} = x^n$ implies, $x^n = xrx^n$, so $R$ is semi $\pi$- regular ring.

$vi \Rightarrow i$ Let $I$ be a proper right ideal and $x \in I$, then there is $r \in R$ and $n \in \mathbb{N}$ such that $x^n = xrx^n$, let $n \geq 2$, then

$$
\begin{aligned}
x^{n-1} - xrx^{n-1}\;^2 = \; x^{n-1} - xrx^{n-1} \;\; x^{n-1} - xrx^{n-1} \\
&= x^{2n-2} - x^n rx^{n-1} - xrx^{2n-2} + xrx^{n-1}xrx^{n-1} \\
&= x^{2n-2} - x^n rx^{n-1} - xrx^n . x^{n-2} + xrx^n . rx^{n-1} \\
&= x^{2n-2} - x^n rx^{n-1} - x^n . x^{n-2} + x^n rx^{n-1} \\
&= 0
\end{aligned}
$$

So $\left(x^{n-1}-xrx^{n-1}\right)\in SN_R=\{0\}$ which implies $x^{n-1}=xrx^{n-1}$. Continuing this process to get $(x-xrx)\in SN_R=\{0\}$ . Thus $x=xrx$ , and since $SN_R=\{0\}$ , then $I_R\subseteq C_R$ (Lemma 2.7). So $x=x\left(xr\right)=xy$ where $y\in I$. Therefore $I$ is pure right ideal and $R$ is fully right pure ring. □

**Theorem 3.6.** *The following are equivalent for any ring $R$ :*

*(i) $R$ is a fully right pure ring.*

*(ii) $R$ is a regular and semi commutative ring.*

*(iii) $R$ is a regular $zi-ring$ .*

*(iv) $R$ is a regular $zc-ring$ .*

**Proof:** $i\Rightarrow ii$ Since $R$ is a fully right pure ring, then it is abelian regular ring (Theorem 3.2). Now, let $x\in aR$ , then $x=ab=(ara)b$ where $ra\in I_R\subseteq C_R$ , so $x=a(ra)b=ab(ra)=(abr)a=\bar{r}a\in Ra$, so $aR\subseteq Ra$, by the same way $Ra\subseteq aR$ . Thus $Ra=aR$ for all $a$ of $R$ and $R$ is a regular and semi commutative ring.

$ii\Rightarrow iii$ Let $ab=0$, then for all $r\in R$ we have $arb=ab\bar{r}$ . Since $R$ is semi commutative ring which implies $arb=0\cdot\bar{r}=0$. So $R$ is regular $zi-ring$ .

$iii\Rightarrow iv$ Let $ab=0$, then we have $ba=(brb)a$ for $R$ is a regular ring , but $R$ is $zi-ring$ , so $br\in I_R\subseteq C_R$ (Lemma 2.5) which implies

$ba=(brb)a=(br)ba=ba(br)=b(ab)r=b(0)r=0$. Thus $R$ is a regular $zc-ring$ .

$iv\Rightarrow i$ Let $I$ be any right ideal of $R$ and $a\in I$ . Since $R$ is regular and $a\in R$ , then there is $b\in R$ such that $a=aba$ but $R$ is $zc-ring$ , so $ab\in I_R\subseteq C_R$ (Lemma 2.7). This implies $a=aba=(ab)a=ea=ae$ , where $e=ab\in I$ . So $I$ is pure right ideal, therefore $R$ is fully right pure ring. □

**Corollary 3.7.** *Let $R$ be a right artinian semi prime and $AC-ring$ , then $R$ is fully right pure ring.*

**Proof:** Since $R$ is semi prime and right artinian, then $R$ is regular by (Lambek 1966, proposition 3.5.2). Since $R$ is regular and $AC-ring$ , then it is fully right pure ring (Theorem 3.5). □

**Corollary 3.8.** *A ring $R$ is fully right pure ring if and only if $R$ is fully completely semi prime right ring and $AC-ring$ .*

**Proof** $\Rightarrow$ Since $R$ is a fully right pure ring, then $R$ is fully semi prime right ring and $AC-ring$ (Theorem 3.5). Since $R$ is $AC-ring$ , then every semi prime right ideal is completely semi prime right ideal (Lemma 2.6). So $R$ is fully completely semi prime right ring and $AC-ring$ .

$\Leftarrow$ Since $R$ is fully completely semi prime right ring, then $R$ is fully semi prime right ring, by assumption $R$ is $AC-ring$, so $R$ is fully right pure ring (Theorem 3.5). □

**Corollary 3.9.** *Let $R$ be a fully right pure ring, then every prime ideal of $R$ is a maximal right ideal of $R$.*
**Proof:** Let $P$ be any prime ideal of $R$. Since $R$ is fully right pure ring, then $R$ is $AC-ring$ (Theorem 3.5), so $P$ is completely prime ideal (Lemma 2.6). Therefore $P$ is a maximal right ideal (Theorem 3.5 (5)). □

**Corollary 3.10.** *If $R$ is a fully right pure ring, then $R$ is duo – ring.*
**Proof:** Let $R$ be a fully right pure ring, so $R$ is strongly regular (Theorem 3.2). Therefore $R$ is regular and $SN_R = \{0\}$ . Since $SN_R = \{0\}$, then $I_R \subseteq C_R$ (Lemma 2.7).
Now,
(i) Let $I$ be any right ideal of $R$ and $a \in I$, then $aR = eR$ for some $e \in I_R$ since $R$ is regular ring, thus $RaR = R\,eR = eR = aR \subseteq I$, which implies that $RaR \subseteq I$ for all $a \in I$. Therefore $I$ is a two sided ideal.
(ii) Let $I$ be a left ideal of $R$ and $a \in I$, then $Ra = \mathrm{Re}$ ; $e \in I_R$, so $RaR = R\,\mathrm{e}R = R\,\mathrm{e} = Ra \subseteq I$, thus $I$ is a two sided ideal. Hence $R$ is duo−ring. □

## 4. Fully Right Pure Group Ring

In this section, we will find the necessary and sufficient conditions on a group $G$ and a ring $A$ for the group ring $A[G]$ to be fully right pure group ring, and we will study some properties of this kind of group rings.

**Theorem 4.1.** (Connell 1963) *A group ring $A[G]$ is a regular ring if and only if:*
*(i) $A$ is a regular ring.*
*(ii) $G$ is a locally finite group, and*
*(iii) The order of any finite subgroup of $G$ is a unit in $A$.*

**Lemma 4.2.** *Let $I$ be any right ideal of a ring $A$ and $G$ be any group. If $I[G]$ is a pure right ideal in $A[G]$, then $I$ is a pure right ideal in $A$.*
**Proof:** Let $a \in I$, then $a \in I[G]$.

But $I[G]$ is a pure right ideal in $A[G]$. So there is $\beta = \sum_{i=1}^{n} b_i g_i$, where $b_i \in I$, $\forall i = 1,2,...,\mathrm{n}$ such that $a = a\beta = a\sum_{i=1}^{n} b_i g_i$, aplay $\delta$ (the augmentation map) on both sides, then we get $a = a\sum_{i=1}^{n} b_i$.

Since $b_i \in I$, $\forall i = 1,2,...,\mathrm{n}$ and $I$ is right ideal in $A$, then $\sum_{i=1}^{n} b_i \in I$

Let $b=\sum_{i=1}^{n} b_i$ , so $a=ab$ where $b \in I$ . Thus $I$ is a pure right ideal in $A$. □

**Theorem 4.3.** *If the group ring $A[G]$ is a fully right pure group ring, then $A$ is a fully right pure ring.*

**Proof:** Let $A[G]$ be a fully right pure group ring and $I$ be any right ideal of $A$ , then $I[G]$ is right ideal of $A[G]$. But $A[G]$ is a fully right pure group ring, so $I[G]$ is a pure right ideal. Hence $I$ is a pure right ideal (Lemma 4.2), therefore $A$ is a fully right pure ring. □

**Theorem 4.4.** *Let $A$ be a commutative ring and $G$ be abelian group, if:*

*(i) $A$ is a regular ring.*

*(ii) $G$ is locally finite group, and*

*(iii) The order of any finite subgroup of $G$ is a unit in $A$ .*

*Then the group ring $A[G]$ is fully pure group ring.*

**Proof:** Since $A$ is commutative and $G$ is abelian, then $A[G]$ is commutative group ring. So the idempotent elements are central.

Since (i) , (ii) and (iii) hold, then (Theorem 4.1) $A[G]$ is regular ring. Thus we have $A[G]$ is an abelian regular ring, so it is fully right pure group ring (Theorem 3.2). □

**Theorem 4.5.** *Let $A$ be a fully right pure ring. If:*

*(i) $G$ is a locally finite abelian group, and*

*(ii) The order of any finite subgroup of $G$ is a unit in $A$ .*

*Then the group ring $A[G]$ is fully right pure group ring.*

**Proof:** Since $A$ is fully right pure ring, then $A$ is abelian regular ring (Theorem 3.2) i.e., $A$ is regular and $I_A \subseteq C_A$ . Since $A$ is a regular ring, and (i) and (ii) hold, then $A[G]$ is regular ring (Theorem 4.1).

Now, let $\alpha=\sum_{i=1}^{n} a_i g_i \in A[G]$, but $A[G]$ is regular, so there is $\beta=\sum_{i=1}^{n} b_i h_i \in A[G]$ such that $\alpha=\alpha\beta\alpha$, i.e. $\sum_{i=1}^{n} a_i g_i=\left(\sum_{i=1}^{n} a_i g_i\right).\left(\sum_{j=1}^{m} b_j h_j\right).\left(\sum_{i=1}^{n} a_i g_i\right)$ . This implies

$$\sum_{i=1}^{n} a_i g_i=\left(\sum_{i=1}^{n} a_i g_i\right).\left[\left(\sum_{j=1}^{m} b_j h_j\right).\left(\sum_{i=1}^{n} a_i g_i\right)\right] \quad \text{this implies}$$

$$\sum_{i=1}^{n} a_i g_i=\left(\sum_{i=1}^{n} a_i g_i\right).\left(\sum_{j=1}^{m} \sum_{i=1}^{n} b_j h_j a_i g_i\right)=\left(\sum_{i=1}^{n} a_i g_i\right).\left(\sum_{j=1}^{m} \sum_{i=1}^{n} b_j a_i h_j g_i\right).$$

But $A$ is regular ring, thus for all $a_i$ there is $c_i$ such that $a_i=a_i c_i a_i$ and $a_i c_i \in I_A \subseteq C_A$ . Therefore

$$\sum_{i=1}^{n} a_i g_i=\left(\sum_{i=1}^{n} a_i g_i\right).\left(\sum_{j=1}^{m} \sum_{i=1}^{n} b_j (a_i c_i a_i) h_j g_i\right)$$

$$=\left(\sum_{i=1}^{n} a_i g_i\right).\left(\sum_{j=1}^{m} \sum_{i=1}^{n} b_j (a_i c_i) a_i h_j g_i\right)$$

$$=\left(\sum_{i=1}^{n} a_i g_i\right)\cdot\left(\sum_{j=1}^{m}\ \sum_{i=1}^{n}(a_i c_i) b_j a_i h_j g_i\right), \text{ since } a_i c_i \in I_A \subseteq C_A .$$

$$=\left(\sum_{i=1}^{n} a_i g_i\right)\cdot\left(\sum_{j=1}^{m}\ \sum_{i=1}^{n}(a_i g_i) c_i b_j a_i h_j\right), \text{ since } G \text{ is abelian.}$$

$$=\left(\sum_{i=1}^{n} a_i g_i\right)\cdot\left(\sum_{i=1}^{n} a_i g_i\right)\cdot\left(\sum_{j=1}^{m}\ \sum_{i=1}^{n} c_i b_j a_i h_j\right).$$

Let $\lambda=\left(\sum_{j=1}^{m}\ \sum_{i=1}^{n} c_i b_j a_i h_j\right)$, then $\alpha=\alpha^2\lambda$, hence $A[G]$ is strongly regular ring, thus $A[G]$ is fully right pure group ring (Theorem 3.2). □

**Theorem 4.6.** *Let $A[G]$ be a regular ring. Then $A[G]$ is fully right pure ring if and if $A[G]$ is a $l.a.r.i-ring$ .*

**Proof:** $\Rightarrow$) $A[G]$ is fully right pure ring, then $A[G]$ is duo−ring (Corollary 3.10), so it is $l.a.r.i.$−ring.

$\Leftarrow$) since $A[G]$ is $l.a.r.i-ring$ then it is $zi-ring$ (Lemma 2.5), therefore $I_{A[G]} \subseteq C_{A[G]}$ (Lemma 2.5). Since $A[G]$ is regular ring by hypothesis, then it is abelian regular, so $A[G]$ is fully right pure ring (Theorem 3.2). □

**Definition 4.7.** A group $G$ is called a Dedekind group if and only if all its subgroups are normal.

**Theorem 4.8.** *Let $A[G]$ be a fully right pure group ring, then $G$ is a Dedekind group.*

**Proof:** Since $A[G]$ is fully right pure group ring, then it is duo−ring (Corollary 3.10).
Now, Let $H$ be any subgroup of $G$. Since $A[G]$ is duo−ring, then $w_G[H]$ is a two sided ideal, so $H$ is normal subgroup of $G$ by (Connell 1963, Proposition 1 (1)), therefore $G$ is a Dedekind group. □

**Corollary 4.9.** *Let $A[G]$ be a fully right pure group ring and $G$ be a non trivial group, then $A[G]$ has at least one non trivial central idempotent.*

**Proof:** Let $G \neq e$ , then there exist $h \in G$ ; $h \neq e$ suppose $H=\langle h\rangle$
Since $G$ is locally finite (Theorem 4.1), then $O(H)=n$ for some $n \in \mathbb{N}$ but $A[G]$ is a regular ring (Theorem 3.2), so $n$ is invertible in $A$ (Theorem 4.1), therefore $e=\frac{1}{n}\sum_{i=0}^{n-1} h^i$ is a non trivial idempotent of $A[G]$ (for if $e=1$ then $n^{-1}+n^{-1}h+n^{-1}h^2+...n^{-1}h^{n-1}-1.e=0$ implies 1= 0 (since the group G forms a base for the $A-$module $A[G]$) which is not true). Also $e$ is central since $A[G]$ is an abelian regular (Theorem 3.2). □

**Proposition 4.10.** *If the group ring* $A[G]$ *is a fully right pure group ring, then the following hold:*

*(i)* $A$ *is a strongly regular ring.*
*(ii)* $A$ *is a regular* $AC-ring$ *.*
*(iii)* $A$ *is a fully idempotent ring.*
*(iv)* $A$ *is a fully semi prime ring.*
*(v)* $A$ *is a semi* $\pi$*- regular ring.*

**Proof:** Directly from Theorem 4.3, Theorem 3.2 and Theorem 3.5. □

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