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AN APPLICATION OF *q*-CALCULUS TO HARMONIC UNIVALENT FUNCTIONS

(Suatu Penggunaan *q*-Kalkulus untuk Fungsi Univalen Harmonik)

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ABSTRACT

The purpose of the present paper is to introduce a new subclass of harmonic univalent functions associated with *q*-calculus. We obtain coefficient conditions, distortion bounds, extreme points, convolution condition and convex combination for functions belonging to this class. We also discuss a class-preserving integral operator and *q*-Jackson type integral operator for this class of functions.

Keywords: harmonic; univalent function; *q*-calculus

ABSTRAK

Makalah ini bertujuan memperkenalkan suatu subkelas fungsi univalen harmonik baharu yang dikaitkan dengan *q*-kalkulus. Diperoleh batas-batas pekali, batas erotan, titik ekstrem, syarat konvolusi dan gabungan cembung untuk fungsi yang terkandung dalam kelas ini. Juga dibincangkan kelas mengekal pengoperasi pengkamir dan pengoperasi pengkamir jenis *q*-Jackson untuk kelas fungsi tersebut.

Keywords: harmonik; fungsi univalen; *q*-kalkulus

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply-connected domain *D* if both *u* and *v* are real harmonic in *D*. In any simply-connected domain we can write $f = h + \overline{g}$, where *h* and *g* are analytic in *D*. We call *h* the analytic part and *g* the co-analytic part of *f*. A necessary and sufficient condition for *f* to be locally univalent and sense-preserving in *D* is that $|h'(z)| > |g'(z)|, z \in D$, see Clunie and Shiel-Small (1984) and Duren (2004).

Let S_H denote the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the open unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions *h* and *g* as,

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \qquad |b_1| < 1. \tag{1}$$

Note that S_H reduces to class *S* of normalized analytic univalent functions if the co-analytic part of its member is zero

Now, we recall the concept of *q*-calculus which may be found in Aral *et al.* (2013).

For $k \in N$, the *q*-number is defined as follows:

$$[k]_q = \frac{1-q^k}{1-q}, \quad 0 < q < 1. \tag{2}$$

Hence, $[k]_q$ can be expressed as a geometric series $\sum_{i=0}^{k-1} q^i$, when $k \to \infty$, then the series converges to $\frac{1}{1-q}$. As $q \to 1$, $[k]_q \to k$ and this is the bookmark of a q-analogue the limit as $q \to 1$ recovers the classical object. The q-derivative of a function f is defined by $D_q(f(z)) = \frac{f(qz) - f(z)}{(q-1)z}, \quad q \neq 1, z \neq 0$, and $D_q(f(0)) = f'(0)$ provided $f'(0)$ exists.

For a function $h(z) = z^k$, we observe that

$$D_q(h(z)) = D_q(z^k) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1}.$$

Then

$$lim_{q \to 1} D_q(h(z)) = lim_{q \to 1}[k]_q z^{k-1} = kz^{k-1} = h'(z),$$

where h' is the ordinary derivative.
The q-Jackson definite integral of the function f is defined by

$$\int_0^z f(t) d_q t = (1-q) z \sum_{n=0}^{\infty} f(zq^n) q^n, \quad z \in \mathbb{C}$$

In 1994 Uralegaddi *et al.* (1994, 1995) introduced the analogues subclasses of starlike, convex and close-to-convex analytic functions with positive coefficients and opened up a new and interesting direction of research in the theory of analytic univalent functions. Motivated with the initial work of Uralegaddi *et al.* (1994, 1995) many researchers (e.g. Dixit and Chandra (2008), Dixit and Pathak (2003), Dixit *et al.* (2013), Porwal and Dixit (2010) and Porwal *et al.* (2011)) introduced and studied various new subclasses of analytic univalent functions with positive coefficients. In 2010, Dixit and Porwal (2010) introduced a new subclass of harmonic univalent function with positive coefficients and opened up a new direction of research in the theory of harmonic univalent functions. After the appearence of this paper, several researchers (e.g. Pathak *et al.* (2012), Porwal and Aouf (2013), Porwal *et al.* (2012)) generalised the result of Dixit and Porwal (2010). Porwal and Dixit (2013) investigated new subclasses of harmonic starlike and convex functions. These results were generalised in Porwal 2015a, Porwal 2015b and 2012.

In the present paper, analogues to the above mentioned works, we introduce a new subclass of harmonic univalent functions by using q-calculus.

Let $M_H[q,\beta]$ denote the family of harmonic functions of the form $f = h + \overline{g}$ satisfying the following condition

$$\Re\left\{\frac{z(zD_q h(z))' - \overline{z(zD_q g(z))'}}{h(z) + \overline{g(z)}}\right\} < \beta \quad \text{for } 1 < \beta \leq \frac{4}{3}, 0 < q < 1. \tag{3}$$

Further, let V_H denote the subclass of S_H consisting of functions of the form,

$$f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \overline{z}^k. \tag{4}$$

Also, we define $V_H(q,\beta) = M_H(q,\beta) \cap V_H$.

In this study, we obtain coefficient bound, extreme points, distortion bounds, convolution, convex combination for functions in the class $V_H(q,\beta)$. Finally we discuss a class preserving integral operator.

2. Main Results

Theorem 2.1. *Let the function $f = h + \overline{g}$ be given by (1). If*

$$\sum_{k=2}^{\infty}\frac{k[k]_q-\beta}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{k[k]_q+\beta}{\beta-1}|b_k|\le 1, \tag{5}$$

then $f\in M_H[q,\beta]$, where $1<\beta\le\frac{4}{3}$ and $0 < q < 1$.

Proof. Let

$$\sum_{k=2}^{\infty}\frac{k[k]_q-\beta}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{k[k]_q+\beta}{\beta-1}\le 1.$$

It suffices to show that, $\left|\dfrac{\dfrac{z(zD_qh(z))'-\overline{z(zD_qg(z))'}}{h(z)+\overline{g(z)}}-1}{\dfrac{z(zD_qh(z))'-z(zD_qg(z))'}{h(z)+\overline{g(z)}}-(2\beta-1)}\right|<1$

$$\le\left|\frac{\dfrac{z+\sum_{k=2}^{\infty}k[k]_qa_kz^k-\overline{\sum_{k=1}^{\infty}k[k]_qb_kz^k}}{z+\sum_{k=2}^{\infty}a_kz^k+\sum_{k=1}^{\infty}\overline{b_kz^k}}-1}{\dfrac{z+\sum_{k=2}^{\infty}k[k]_qa_kz^k-\sum_{k=1}^{\infty}k[k]_qb_kz^k}{z+\sum_{k=2}^{\infty}a_kz^k+\sum_{k=1}^{\infty}\overline{b_kz^k}}-(2\beta-1)}\right|$$

$$\le\frac{\sum_{k=2}^{\infty}(k[k]_q-1)|a_k||z|^{k-1}+\sum_{k=1}^{\infty}(k[k]_q+1)|b_k||z|^{k-1}}{2(\beta-1)-\sum_{k=2}^{\infty}(k[k]_q-2\beta+1)|a_k||z|^{k-1}-\sum_{k=1}^{\infty}(k[k]_q+2\beta-1)|b_k||z|^{k-1}}.$$

The last expression is bounded above by 1, if

$$\sum_{k=2}^{\infty}(k[k]_q-1)|a_k|+\sum_{k=1}^{\infty}(k[k]_q+1)|b_k|\leq 2(\beta-1)-\sum_{k=2}^{\infty}(k[k]_q-2\beta+1)|a_k|-\sum_{k=1}^{\infty}(k[k]_q+2\beta-1)|b_k|,$$

$$\sum_{k=2}^{\infty}(k[k]_q-1)|a_k|+\sum_{k=2}^{\infty}(k[k]_q-2\beta+1)|a_k|+\sum(k[k]_q+1+[k]_q+2\beta-1)|b_k|\leq 2(\beta-1),$$

$$2\sum_{k=2}^{\infty}(k[k]_q-\beta)|a_k|+2\sum_{k=1}^{\infty}(k[k]_q+\beta|b_k|\leq 2(\beta-1),$$

equivalent to

$$\sum_{k=2}^{\infty}\frac{k[k]_q-\beta}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{k[k]_q+\beta}{\beta-1}|b_k|\leq 1.$$

Hence

$$\left|\frac{\dfrac{z(zD_qh(z))'-\overline{z(zD_qg(z))'}}{h(z)+\overline{g(z)}}-1}{\dfrac{z(zD_qh(z))'-\overline{z(zD_qg(z))'}}{h(z)+\overline{g(z)}}-(2\beta-1)}\right|<1,\ z\in U.$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. *A function of the form (4) is in* $V_H(q,\beta)$, *if and only if,*

$$\sum_{k=2}^{\infty}\frac{k[k]_q-\beta}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{k[k]_q+\beta}{\beta-1}|b_k|\leq 1. \quad (6)$$

Proof. Since $V_H(q,\beta)\subset M_H(q,\beta)$, the "if" part follows from Theorem 2.1. For "only if" part we show that $f\in V_H(q,\beta)$ if the above condition does not hold.

Note that a neccessary and sufficient condition for $f=h+\overline{g}$ given by (4) is in $V_H(q,\beta)$, if,

$$\Re\left\{\frac{z(zD_qh(z))'-\overline{z(zD_qg(z))'}}{h(z)+\overline{g(z)}}\right\}<\beta,$$

is equivalent to

$$\Re\left\{\frac{(\beta-1)z-\sum_{k=2}^{\infty}(k[k]_q-\beta)|a_k|z^k-\sum_{k=1}^{\infty}(k[k]_q+\beta)|b_k|\overline{z^k}}{z+\sum_{k=2}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\overline{z^k}}\right\}\geq 0.$$

The above condition must hold for all values of z, $|z|=r<1$, upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$\left\{\frac{(\beta-1)-\sum_{k=2}^{\infty}(k[k]_q-\beta)|a_k|r^{k-1}-\sum_{k=1}^{\infty}(k[k]_q+\beta)|b_k|r^{k-1}}{1+\sum_{k=2}^{\infty}|a_k|r^{k-1}-\sum_{k=1}^{\infty}|b_k|r^{k-1}}\right\}\geq 0. \tag{7}$$

If the condition (6) does not hold then the numerator of (7) is negative for r sufficiently close to 1. Thus there exist a $z_0=r_0$ in (0, 1) for which the quotient in (7) is negative. This contradicts with the required condition for $f\in V_H(q,\beta)$ and so the proof is completed. □

Next, we determine the extreme points of the closed convex hulls of $V_H(q,\beta)$, denoted by clco $V_H(q,\beta)$.

Theorem 2.3. *A function $f\in clcoV_H(q,\beta)$, if and only if*

$$f(z)=\sum_{k=1}^{\infty}\{x_k h_k(z)+y_k g_k(z)\}, \tag{8}$$

where $h_1(z)=z,\ h_k(z)=z+\frac{\beta-1}{k[k]_q-\beta}z^k,\quad (k=2,3,)$ *and* $g_k(z)=z-\frac{\beta-1}{k[k]_q+\beta}\overline{z}^k,$

(k = 1, 2, ...), $\sum_{k=1}^{\infty}(x_k+y_k)=1,\quad x_k\geq 0,\ y_k\geq 0.$

In particular the extreme points of $V_H(q,\beta)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. Suppose that

$$f(z)=\sum_{k=2}^{\infty}\{x_k h_k(z)+y_k g_k(z)\}$$

$$=z+\sum_{k=2}^{\infty}\frac{\beta-1}{k[k]_q-\beta}x_k z^k-\sum_{k=1}^{\infty}\frac{\beta-1}{k[k]_q+\beta}y_k\overline{z^k}.$$

Then,

$$\sum_{k=2}^{\infty}\frac{k[k]_q-\beta}{\beta-1}\left\{\frac{\beta-1}{k[k]_q-\beta}x_k\right\}+\sum_{k=1}^{\infty}\frac{k[k]_q+\beta}{\beta-1}\left\{\frac{\beta-1}{k[k]_q+\beta}y_k\right\}=\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k=1-x_1\leq 1,$$

and so $f\in clcoV_H[q,\beta]$.

Conversely if $f\in clcoV_H[q,\beta]$, set

$$x_k=\frac{k[k]_q-\beta}{\beta-1}|a_k|,\ k=2,3,4,\ldots\text{and } y_k=\frac{k[k]_q+\beta}{\beta-1}|b_k|,\ k=1,2,3,\ldots$$

from Theorem 2.2 we have, $0 \le x_k \le 1,$ and $0 \le y_k \le 1, (k = 1,2,3,\ldots).$ We define $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$ and by Theorem 2.2, $x_1 \ge 0.$

Consequently, we obtain $f(z) = \sum_{k=1}^{\infty} \{x_k h_k(z) + y_k g_k(z)\}$ as required. □

Theorem 2.4. *Let* $f \in V_H(q,\beta)$. *Then for* $|z| = r < 1$, *we have,*

$$|f(z)| \le (1+|b_1|)r + \left(\frac{\beta-1}{2[2]_q - \beta} - \frac{\beta+1}{2[2]_q - \beta} |b_1| \right) r^2$$

and

$$|f(z)| \ge (1-|b_1|)r - \left(\frac{\beta-1}{2[2]_q - \beta} - \frac{\beta+1}{2[2]_q - \beta} |b_1| \right) r^2.$$

Proof. We only prove the right hand inequality. The proof for left hand inequality is similar and will be omitted. Let $f \in V_H(q,\beta)$, taking the absolute value of *f*, we have,

$$\begin{aligned}
|f(z)| &\le (1+|b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k, \\
&\le (1+|b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^2, \\
&= (1+|b_1|)r + r^2 \sum_{k=2}^{\infty} (|a_k| + |b_k|), \\
&= (1+|b_1|)r + r^2 \frac{\beta-1}{2[2]_q - \beta} \sum_{k=2}^{\infty} \frac{2[2]_q - \beta}{\beta - 1} (|a_k| + |b_k|), \\
&= (1+|b_1|)r + r^2 \frac{\beta-1}{2[2]_q - \beta} \sum_{k=2}^{\infty} \frac{k[k]_q - \beta}{\beta - 1} (|a_k| + |b_k|), \\
&\le (1+|b_1|)r + r^2 \frac{\beta-1}{2[2]_q - \beta} \sum_{k=2}^{\infty} \left(\frac{k[k]_q - \beta}{\beta - 1} |a_k| + \frac{k[k]_q + \beta}{\beta - 1} |b_k| \right), \\
&= (1+|b_1|)r + r^2 \frac{\beta-1}{2[2]_q - \beta} \left(1 - \frac{1+\beta}{\beta-1} |b_1| \right), \\
&= (1+|b_1|)r + r^2 \left(\frac{\beta-1}{2[2]_q - \beta} - \frac{1+\beta}{2[2]_q - \beta} |b_1| \right).
\end{aligned}$$

Thus the proof of Theorem 2.4 is established. □

Theorem 2.5. *Let* $f \in V_H(q,\alpha)$ *and* $F \in V_H(q,\beta)$. *Then* $f * F \in V_H(q,\alpha) \subseteq V_H(q,\beta)$, *for* $1 < \alpha \le \beta \le \frac{4}{3}$ *and* $0 < q < 1$.

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \bar{z}^k$ be in $V_H(q,\alpha)$ and $F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k - \sum_{k=1}^{\infty} |B_k| \bar{z}^k$ be in $V_H(q,\beta)$.

Then the convolution $f * F$ is given by

$$(f * F)(z) = f(z) * F(z) = z + \sum_{k=2}^{\infty} |a_k A_k| z^k - \sum_{k=1}^{\infty} |b_k B_k| \bar{z}^k.$$

We wish to show that the coefficients of $f * F$ satisfy the required condition in Theorem 2.2 for $F(z) \in V_H(q,\beta)$. We note that $|A_k| \le 1$ and $|B_k| \le 1$. Now for the convolution function $f * F$, we obtain,

$$\sum_{k=2}^{\infty} \frac{k[k]_q - \beta}{\beta - 1} |a_k A_k| + \sum_{k=1}^{\infty} \frac{k[k]_q + \beta}{\beta - 1} |b_k B_k|$$

$$\le \sum_{k=2}^{\infty} \frac{k[k]_q - \beta}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k[k]_q + \beta}{\beta - 1} |b_k|$$

$$\le 1, \quad \text{since } f(z) \in V_H(q,\beta).$$

Therefore $f * F \in V_H(q,\alpha) \subseteq V_H(q,\beta)$.
Thus the proof of Theorem 2.5 is established. □

Theorem 2.6. *The class* $V_H(q,\beta)$ *is closed under convex combination.*

Proof. Let $f_i(z) \in V_H(q,\beta)$, $i = \{1,2,3,\ldots\}$, where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{k=2}^{\infty} |a_{ki}| z^k - \sum_{k=1}^{\infty} |b_{ki}| \bar{z}^k.$$

Then, by Theorem 2.2,

$$\sum_{k=2}^{\infty} \frac{k[k]_q - \beta}{\beta - 1} |a_{ki}| + \sum_{k=1}^{\infty} \frac{k[k]_q + \beta}{\beta - 1} |b_{ki}| \le 1. \tag{9}$$

The convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{ki}| \right) z^k - \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{ki}| \right) \bar{z}^k, \text{ for } \sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1.$$

Then by Theorem 2.2, we have,

$$\sum_{k=2}^{\infty}\frac{k[k]_q-\beta}{\beta-1}\left(\sum_{i=1}^{\infty}t_i\,|a_{ki}|\right)+\sum_{k=1}^{\infty}\frac{k[k]_q+\beta}{\beta-1}\left(\sum_{i=1}^{\infty}t_i\,|b_{ki}|\right)$$
$$=\sum_{i=1}^{\infty}t_i\left(\sum_{k=2}^{\infty}\frac{k[k]_q-\beta}{\beta-1}|a_{ki}|+\sum_{k=1}^{\infty}\frac{k[k]_q+\beta}{\beta-1}|b_{ki}|\right)$$
$$\le\sum_{i=1}^{\infty}t_i=1.$$

This is the condition required by Theorem 2.2, and hence we have $\sum_{i=1}^{\infty}t_if_i(z)\in V_H(q,\beta)$.

The proof of Theorem 2.6 is completed. □

3. A Family of Class Preserving Integral Operator

Let $f(z)$ be defined by (1). Then let us define *F(z)* by the relation,

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt+\overline{\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt},\quad (c>-1). \tag{10}$$

Theorem 3.1. *Let* $f(z)=h(z)+\overline{g(z)}\in S_H$ *be given by* (4) *and* $f\in V_H(q,\beta)$ *, where* $1<\beta\le\frac{4}{3}$. *Then F(z) defined by* (10) *is also in the class* $V_H(q,\beta)$.

Proof. Let $f(z)=z+\sum_{k=2}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\overline{z}^k$ be in $V_H(q,\beta)$. Then by Theorem 2.2, we have,

$$\sum_{k=2}^{\infty}\frac{k[k]_q-\beta}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{k[k]_q+\beta}{\beta-1}|b_k|\le 1.$$

From the equation (10) of *F(z)*, it follows that,

$$F(z)=z+\sum_{k=2}^{\infty}\frac{c+1}{c+k}|a_k|z^k-\sum_{k=1}^{\infty}\frac{c+1}{c+k}|b_k|\overline{z}^k.$$

Now,

$$\sum_{k=2}^{\infty}\frac{k[k]_q-\beta}{\beta-1}\left(\frac{c+1}{c+k}\right)|a_k|+\sum_{k=1}^{\infty}\frac{k[k]_q-\beta}{\beta-1}\left(\frac{c+1}{c+k}\right)|b_k|$$
$$\le\sum_{k=2}^{\infty}\frac{k[k]_q-\beta}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{k[k]_q+\beta}{\beta-1}|b_k|$$
$$\le 1.$$

Thus $F(z)\in V_H(q,\beta)$.

The proof of Theorem 3.1 is completed. □

Definition 3.1. Let $f = h + \overline{g}$ be defined by (1). Then the q-Jackson-type integral operator $F_q : H \to H$ is defined by the relation,

$$F_q(z) = \frac{[c+1]_q}{z^c}\int_0^z t^{c-1}h(t)d_qt + \overline{\frac{[c+1]_q}{z^c}\int_0^z t^{c-1}g(t)d_qt}, \quad (c > -1), \tag{11}$$

where $[a]_q$ is the q-number defined by (2) and H is the class of functions of the form (1) which are harmonic in U.

Theorem 3.2. *Let* $f(z) = h(z) + \overline{g(z)}$ *be given by* (4) *and* $f \in V_H(q,\beta)$ *where* $1 < \beta \le \frac{4}{3}$, $0 < q < 1$. *Then* $F_q(z)$ *defined by* (11) *is also in the class* $V_H(q,\beta)$.

Proof. Let $f(z) = z + \sum_{k=2}^{\infty}|a_k|z^k - \sum_{k=1}^{\infty}|b_k|\overline{z^k}$ be in $V_H(q,\beta)$. Then by Theorem 2.2, we have $\sum_{k=2}^{\infty}\frac{k[k]_q - \beta}{\beta - 1}|a_k| + \sum_{k=1}^{\infty}\frac{k[k]_q + \beta}{\beta - 1}|b_k| \le 1.$

From the representation (11) of $F_q(z)$, it follows that,

$$F_q(z) = z + \sum_{k=2}^{\infty}\frac{[c+1]_q}{[k+c]_q}|a_k|z^k - \sum_{k=1}^{\infty}\frac{[c+1]_q}{[k+c]_q}|b_k|\overline{z^k}.$$

Since

$$[\mathrm{k}+\mathrm{c}]_q - [\mathrm{c}+1]_q = \sum_{i=0}^{k+c-1}q^i - \sum_{i=0}^{c}q^i = \sum_{i=c+1}^{k+c}q^i > 0$$

$$[\mathrm{k}+\mathrm{c}]_q > [\mathrm{c}+1]_q$$

or $\quad \frac{[c+1]_q}{[k+c]_q} < 1.$

Now

$$\sum_{k=2}^{\infty}\frac{k[k]_q - \beta}{\beta - 1}\frac{[c+1]_q}{[k+c]_q}|a_k| + \sum_{k=1}^{\infty}\frac{k[k]_q + \beta}{\beta - 1}\frac{[c+1]_q}{[k+c]_q}|b_k|$$

$$\le \sum_{k=2}^{\infty}\frac{k[k]_q - \beta}{\beta - 1}|a_k| + \sum_{k=1}^{\infty}\frac{k[k]_q + \beta}{\beta - 1}|b_k|$$

$$\le 1.$$

Thus the proof of Theorem 3.2 is established. □

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