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NEW OSCILLATION CONDITIONS FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

(Syarat Baharu Ayunan bagi Persamaan Pembezaan Tak Linear Peringkat Kedua)

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ABSTRACT

New oscillation conditions for the nonlinear second order differential equations are presented. These criteria involve the use of integral averaging technique. Theorems created in this study are stated in general form, which complement and lengthen some related results known in the literature. The importance of our results is to show that some of the results from previous researches contain superfluous conditions.

Keywords: second order; nonlinear differential equations; oscillation criteria; averaging technique

ABSTRAK

Syarat baharu ayunan bagi persamaan pembezaan tak linear peringkat kedua dikemukakan. Kriterium ini melibatkan penggunaan teknik pemurataan kamiran. Teorem yang dihasilkan dinyatakan dalam bentuk umum yang ia melengkapi dan memperluaskan hasil terdahulu dalam kesusasteraan. Kepentingan hasil yang diperoleh menunjukkan terdapat hasil-hasil daripada kajian lepas yang mengandungi syarat-syarat yang tidak perlu.

Kata kunci: peringkat kedua; persamaan pembezaan tak linear; kriterium ayunan; teknik pemurataan

1. Introduction

The theory of oscillatory is one of the important parts of the qualitative theory of differential equations. Great deals of work on the behaviour of this topic has been developed rapidly in the last decades. (Agarwal *et al*. 2010; Beqiri & Koci 2012; Remili 2010; Salhin *et al*. 2014; Temtek & Tiryaki 2013; Yibing *et al*. 2013; Zhang & Wang 2010). Remili (2008) studied the equation

$$(r(t)x'(t))' + Q(t,x) = H[t, x'(t), x(t)], \tag{1}$$

and successfully derived some oscillation criteria for Equation (1). New results with additional appropriate weighted function have been investigated. Zhang and Wang (2010), studied the following equation,

$$[r(t)\psi(x(t))x'(t)]' + Q(t,x) = H[t, x'(t), x(t)]. \tag{2}$$

Temtek and Tiryaki (2013) achieved a number of new oscillation results for the equation below:

$$[r(t)\psi(x(t))\left|x'(t)\right|^{\alpha-1} x'(t)]' + Q(t,x) = H[t, x'(t), x(t)], \tag{3}$$

and its special cases by using generalized Riccati transformation and well known techniques.

In this paper, we consider the oscillatory of solutions for the nonlinear second order differential equation

$$\left[r(t)\psi(x(t))f(x'(t)))' + q(t)g(x(t)\right] = H[t, x'(t), x(t)], \tag{4}$$

where $r(t)$ is a continuous positive function and q is a continuous functions on $[t_0, \infty)$, $t_0 \geq 0$. $\psi(x) > 0, \forall x \in \mathbb{R}$, where ψ and f are continuous functions on the real line $\mathbb{R}$. g is a continuous differentiable function on the real line $\mathbb{R}$, except possible at 0 with $xg(x) > 0$ and $g'(x) \geq k > 0$ for all $x \neq 0$ and k is a constant. H is a continuous function on $[t_0, \infty) \times \mathbb{R}^2$ with $\frac{H(t, y, x)}{g(x)} \leq p(t)$, $\forall\, t \in [t_0, \infty), y \in \mathbb{R}$ and $x \neq 0$.

A solution $x(t)$ is said to be oscillatory if it has a sequence of zero clustering at ∞ and nonoscillatory otherwise. Thus a nonoscillatory is either eventually positive or eventually negative. Eq. (4) is called oscillatory if all of its solutions are oscillatory and otherwise it is called nonoscillatory. Lots of work have been done on the following particular cases of Eq. (4) (Baculikova 2006; Elabbasy & Elsharabasy 1997; Graef *et al*. 1978; Remili 2008; Tiryaki & Ayanlar 2004; Wintner 1949; Yan 1986; Yeh 1982) such as

$$(r(t)(x'(t))^{\alpha})' + q(t)g(x(t)) = H(t), \tag{5}$$

$$\left(r(t)\psi(x(t))x'(t)\right)' + q(t)g(x(t)) = H(t), \tag{6}$$

$$\left(r(t)x'(t)\right)' + Q(t, x(t)) = H(t, x(t), x'(t)), \tag{7}$$

$$\left(r(t)\psi(x(t))f(x'(t))\right)' + q(t)g(x(t)) = H(t). \tag{8}$$

The averaging technique is an important tool in the study of the oscillatory behaviour of the solution for nonlinear second order differential equations. In this paper, the previous equations given by Baculikova (2006) and Tiryaki *et al*. (2004) are expanded. Previous oscillatory conditions constructed by Elabbasy and Elsharabasy (1997), Graef *et al*. (1978) and Remili (2008) are also modified and expanded.

2. Main Results

In this section, the Riccati's technique is used to establish sufficient conditions for Eq. (4) to be oscillatory.

Theorem 2.1. *Suppose that*

(A_1) $\frac{g'(x)}{\psi(x)} \geq K > 0$ *for a constant K and* $x \neq 0$,

(A_2) $\int_{\pm\varepsilon}^{\pm\infty} \frac{\psi(y)}{g(y)} dy < \infty$ *for all* $\varepsilon > 0$,

(A_3) $0 < k_1 \le \dfrac{f(y)}{y} \le k_2$ *for constants* k_1, k_2, $y = x'(t) \neq 0$ *and* $\lim\limits_{t\to 0} \dfrac{f(y)}{y}$ *exist.*

There also exists a positive function $\rho \in C^1[t_0,\infty)$ *such that* $\left(\rho(t)r(t)\right)' \le 0$ *for all* $t \ge t_0$*, and*

$$(\mathrm{A}_4)\ \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left[\rho(u)[q(u)-p(u)] - \frac{1}{4M}\frac{(\rho'(u))^2}{\rho(u)} r(u)\right] du\,ds = \infty; \quad M = \frac{K}{k_2}.$$

Then Eq. (4) is oscillatory.

Proof. On the contrary assume that Eq. (4) has a nonoscillatory solution $x(t)$. We suppose without loss of generality that $x(t) > 0$ for all $t \in [t_0,\infty)$. Define the function $w(t)$ as

$$w(t) = \frac{\rho(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))} \quad \text{for all} \quad t \ge t_0. \tag{9}$$

Differentiating Eq. (9) and substituting Eq. (4) implies

$$\begin{aligned} w'(t) &= \frac{\rho(t)\left[r(t)\psi(x(t))f(x'(t))\right]'}{g(x(t))} + \frac{\rho'(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))} \\ &\quad - \frac{\rho(t)r(t)\psi(x(t))f(x'(t))g'(x(t))x'(t))}{g^2(x(t))}, \\ &= \frac{\rho(t)H(t,x'(t),x(t))}{g(x(t))} - \rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) \\ &\quad - \frac{1}{\rho(t)r(t)\psi(x(t))f(x'(t))}x'(t)g'(x(t))w^2(t). \end{aligned}$$

From conditions (A_1) and (A_3), we obtain

$$w'(t) \le \rho(t)\left(p(t)-q(t)\right) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{K}{\rho(t)r(t)k_2}w^2(t),$$

$$\rho(t)\left(q(t)-p(t)\right) \le \frac{\rho'(t)}{\rho(t)}w(t) - \frac{M}{\rho(t)r(t)}w^2(t) - w'(t).$$

Integrating the above inequality from t_0 to t gives the following result

$$\int_{t_0}^{t}\rho(s)\left(q(s)-p(s)\right)ds \le w(t_0) - w(t) - \int_{t_0}^{t}\left(\frac{M}{r(s)\rho(s)}w^2(s) - \frac{\rho'(s)}{\rho(s)}w(s)\right)ds,$$

$$\int_{t_0}^{t}\left[\rho(s)\big(q(s)-p(s)\big)-\frac{r(s)\rho(s)}{4M}\left(\frac{\rho'(s)}{\rho(s)}\right)^2\right]ds$$
$$\le\ w(t_0)-w(t)-\int_{t_0}^{t}\left[\sqrt{\frac{M}{r(s)\rho(s)}}w(s)-\frac{\sqrt{r(s)\rho(s)}}{2\sqrt{M}}\frac{\rho'(s)}{\rho(s)}\right]^2 ds,$$
$$\le w(t_0)-w(t).$$

or

$$\int_{t_0}^{t}\left[\rho(s)\big(q(s)-p(s)\big)-\frac{r(s)\rho(s)}{4M}\left(\frac{\rho'(s)}{\rho(s)}\right)^2\right]ds\le w(t_0)-\frac{\rho(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))}.$$

Using condition (A_3),

$$\int_{t_0}^{t}\left[\rho(s)\big(q(s)-p(s)\big)-\frac{r(s)\rho(s)}{4M}\left(\frac{\rho'(s)}{\rho(s)}\right)^2\right]ds\le w(t_0)-k_1\frac{\rho(t)r(t)\psi(x(t))x'(t)}{g(x(t))}.$$

Taking a second integration from t_0 to t gives

$$\int_{t_0}^{t}\int_{t_0}^{s}\left[\rho(u)\big(q(u)-p(u)\big)-\frac{r(u)}{4M}\frac{(\rho'(u))^2}{\rho(u)}\right]du\,ds\le w(t_0)(t-t_0)-k_1\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))x'(s)}{g(x(s))}ds. \quad (10)$$

Since $r(t)\rho(t)$ is nonincreasing, then by the Bonnet's Theorem (Bartle 1976) there exists an $\eta\in[t_0,t]$ such that

$$-k_4\int_{t_0}^{t}\rho(s)r(s)\frac{\psi(x(s))x'(s)}{g(x(s))}ds=-k_1r(t_0)\rho(t_0)\int_{t_0}^{\eta}\frac{\psi(x(s))x'(s)}{g(x(s))}ds,$$
$$=k_1r(t_0)\rho(t_0)\int_{x(\eta)}^{x(t_0)}\frac{\psi(y)}{g(y)}dy,$$
$$<\begin{cases}0, & \text{if } x(t_0)<x(\eta),\\ k_1r(t_0)\rho(t_0)\int_{\varepsilon}^{\infty}\frac{\psi(y)}{g(y)}dy & \text{if } x(t_0)>x(\eta),\end{cases}$$

hence

$$-\infty<-k_1\int_{t_0}^{t}r(s)\rho(s)\frac{\psi(x)x'(s)}{g(x)}ds<L_1,$$

where $L_1=k_1r(t_0)\rho(t_0)\int_{\varepsilon}^{\infty}\frac{\psi(y)}{g(y)}dy$.

Therefore (10) becomes

$$\int_{t_0}^{t}\int_{t_0}^{s}\left[\rho(u)\big(q(u)-p(u)\big)-\frac{r(u)}{4M}\frac{(\rho'(u))^2}{\rho(u)}\right]du\,ds \le w(t_0)(t-t_0)+L_1. \tag{11}$$

Dividing (11) by t and taking the upper limit as $t\to\infty$,

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left[\rho(u)\big(q(u)-p(u)\big)-\frac{r(u)}{4M}\frac{(\rho'(u))^2}{\rho(u)}\right]du\,ds \le \limsup_{t\to\infty}\frac{1}{t}\big[w(t_0)(t-t_0)+L_1\big].$$

$$<\infty.$$

This contradicts assumption (A_4), which completes the proof. □

Example 2.1. Consider the following equation

$$\left[\frac{1}{t}\left(3x'(t)+\frac{(x'(t))^3}{(x'(t))^2+1}\right)\right]'+\left(\frac{1}{2}+\sin t\right)x^3(t)=\frac{x^7(t)\cos t\sin x'(t)}{\left(x^4(t)+1\right)t^3},\quad t>0.$$

Notice that

$$\frac{H(t,x'(t),x(t))}{g(x(t))}=\frac{x^7\cos t\sin x'}{(x^4+1)t^3}\times\frac{1}{x^3}\le\frac{1}{t^3}=p(t),\quad \forall x',x\in\mathbb{R}\text{ and } t\ge t_0,$$

and

$$\frac{g'(x)}{\psi(x)}=\frac{3x^2}{1}\ge 3=k,\qquad \forall\, x\ne 0.$$

For all $\varepsilon>0$, we get

$$\int_{\pm\varepsilon}^{\pm\infty}\frac{\psi(y)}{g(y)}\,dy=-\frac{1}{2y^2}\Bigg|_{\pm\varepsilon}^{\pm\infty}<\infty.$$

Since $f(y)=3y+\dfrac{y^3}{y^2+1}$ so,

$$3<\frac{f(y)}{y}=3+\frac{y^2}{y^2+1}<4,\ \forall y\ne 0,\ \lim_{y\to 0}\frac{f(y)}{y}=3.$$

Let

$$\rho(t) = 1 \Rightarrow \rho'(t) = 0 \in C^1\left[t_0, \infty\right),$$

$$\left(\rho(t)r(t)\right)' = -\frac{1}{t^2} < 0,$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left[\rho(u)\left(q(u)-p(u)\right) - \frac{r(u)\rho(u)}{4M}\frac{(\rho'(s))^2}{\rho(s)}\right]du\ ds$$

$$= \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left[1\left[\frac{1}{2}+\sin u-\frac{1}{u^3}\right]-\frac{\frac{1}{u}\times 1}{4M}\times 0\right]du\, ds$$

$$= \infty.$$

Thus, the theorem ensures that every solution from this example is oscillatory.

Remark 2.1. Theorem 2.1 improves and complements Theorem 4 of Graef *et al*. (1978) and Theorem 1 of Elabbasy and Elsharabasy (1997).

Theorem 2.2. *Assume that* (A_2) *holds and*

$$\left(\mathrm{A}_5\right)\quad \frac{f(y)}{y} > L > 0; \text{ for a constant } L \text{ and } y \neq 0,$$

then there exists a positive continuously differentiable function ρ *as defined in Theorem 2.1 and* $\int_{t_0}^{\infty}\rho(s)ds = \infty.$ *Then Eq.*(4) *is oscillatory if*

$$\left(\mathrm{A}_6\right)\ \limsup_{t\to\infty}\left[\int_{t_0}^{t}\rho(s)ds\right]^{-1}\int_{t_0}^{t}\rho(s)\left\{\int_{t_0}^{s}\left[q(u)-p(u)\right]du\right\}ds = \infty.$$

Proof. On the contrary we assume that Eq. (4) has a nonoscillatory solution $x(t)$. We suppose without loss of generality that $x(t) > 0$ for all $t \in [t_0, \infty)$. Define the function $w(t)$ as

$$w(t) = \rho(t)\int_{t_0}^{t}\frac{r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds \text{ for all } t \geq t_0.$$

Thus, for every $t \geq t_0$ we obtain

$$w'(t) = \rho'(t)\int_{t_0}^{t} \frac{r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds + \frac{\rho(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))}. \tag{12}$$

From Eq. (4), we have

$$\frac{\left[r(t)\psi(x(t))f(x'(t))\right]'}{g(x(t))} \le p(t) - q(t).$$

Integrating the above inequality from t_0 to t and integrate the left hand side by parts we obtain,

$$\frac{r(t)\psi(x(t))f(x'(t))}{g(x(t))} - B + \int_{t_0}^{t} \frac{r(s)\psi(x(s))f(x'(s))g'(x(s))x'(s)}{g^2(x(s))}ds$$
$$\le -\int_{t_0}^{t} q(s) - p(s)ds,$$

where

$$B = \frac{r(t_0)\psi(x(t_0))f(x'(t_0))}{g(x(t_0))},$$

or

$$\frac{r(t)\psi(x(t))f(x'(t))}{g(x(t))} - B \le -\int_{t_0}^{t} q(s) - p(s)ds. \tag{13}$$

Now, multiply the last inequality by $\rho(t)$ we obtain,

$$\frac{\rho(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))} \le B\rho(t) - \rho(t)\int_{t_0}^{t} q(s) - p(s)ds. \tag{14}$$

Substituting (14) in (12) yields

$$w'(t) - \rho'(t)\int_{t_0}^{t} \frac{r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds \le \rho(t)B - \rho(t)\int_{t_0}^{t} (q(s) - p(s))ds.$$

Integrate above inequality from t_0 to t we get

$$\begin{aligned}\int_{t_0}^{t} \left\{ \rho(s)\int_{t_0}^{s} (q(u) - p(u))du \right\} ds \le & w(t_0) - w(t) + B\int_{t_0}^{t} \rho(s)ds \\ & + \int_{t_0}^{t} \left\{ \rho'(s)\int_{t_0}^{s} \frac{r(u)\psi(x(u))f(x'(u))}{g(x(u))}du \right\} ds.\end{aligned} \tag{15}$$

Now evaluate by parts the integral

$$\int_{t_0}^{t}\left\{\rho'(s)\int_{t_0}^{s}\frac{r(u)\psi(x(u))f(x'(u))}{g(x(u))}du\right\}ds=\rho(s)\int_{t_0}^{s}\frac{r(u)\psi(x(u))f(x'(u))}{g(x(u))}du\Bigg|_{t_0}^{t}$$

$$-\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds,$$

$$=\rho(t)\int_{t_0}^{t}\frac{r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds-\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds$$

$$=w(t)-\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds. \tag{16}$$

Substituting (16) in (15) we obtain

$$\int_{t_0}^{t}\left\{\rho(s)\int_{t_0}^{s}(q(u)-p(u))du\right\}ds$$

$$\leq w(t_0)-w(t)+B\int_{t_0}^{t}\rho(s)ds+w(t)-\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds$$

$$\leq w(t_0)+B\int_{t_0}^{t}\rho(s)ds-\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))f(x'(s))}{g(x(s))}ds.$$

From condition (A_5) we get

$$\int_{t_0}^{t}\left\{\rho(s)\int_{t_0}^{s}(q(u)-p(u))du\right\}ds\leq w(t_0)+B\int_{t_0}^{t}\rho(s)ds-L\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))x'(s)}{g(x(s))}ds. \tag{17}$$

Since $(\rho(t)r(t))'\leq 0$ then by Bonnet's Theorem (Bartle 1976) there exist $\eta\in[t_0,t]$ such that

$$-\int_{t_0}^{t}\frac{\rho(s)r(s)\psi(x(s))x'(s)}{g(x(s))}ds=-\rho(t_0)r(t_0)\int_{t_0}^{\eta}\frac{\psi(x(s))x'(s)}{g(x(s))}ds$$

$$=\rho(t_0)r(t_0)\int_{x(\eta)}^{x(t_0)}\frac{\psi(y)}{g(y)}dy$$

$$<\begin{cases}0 & \text{if } x(t_0)<x(\eta),\\ L\,r(t_0)\rho(t_0)\int_{\varepsilon}^{\infty}\frac{\psi(y)}{g(y)}dy & \text{if } x(t_0)>x(\eta).\end{cases}$$

Hence

$$-\infty < -\int_{t_0}^{t} \frac{\rho(s) r(s) \psi(x(s)) x'(s)}{g(x(s))} ds \le B_1,$$

where

$$B_1 = \rho(t_0) r(t_0) \int_{x(\eta)}^{x(t_0)} \frac{\psi(y)}{g(y)} dy.$$

Consequently,

$$\int_{t_0}^{t} \left\{ \rho(s) \int_{t_0}^{s} (q(u) - p(u)) du \right\} ds \le M + B \int_{t_0}^{t} \rho(s) ds,$$

where

$$M = w(t_0) + LB_1.$$

Then

$$\left[\int_{t_0}^{t} \rho(s) ds \right]^{-1} \int_{t_0}^{t} \left\{ \rho(s) \int_{t_0}^{s} (q(u) - p(u)) du \right\} ds \le \frac{M}{\int_{t_0}^{t} \rho(s) ds} + B.$$

By taking the upper limit, as $t \to \infty$, this contradicts the assumption (A_6). The proof is completed. □

Example 2.2. Consider the nonlinear differential equation

$$\left[\frac{1}{t} x^2(t) x'(t) \right]' + t^3 x(t) \left(1 + x^2(t)\right)^2 = \frac{x^3(t)}{\left(x^2(t) + 1\right)^2} \sin t \frac{(x'(t))^2}{(x'(t))^2 + 1}, \quad t \ge 1.$$

Note that

$$\begin{aligned} \frac{H(t, x'(t), x(t))}{g(x(t))} &= x^3(t) \sin t \frac{(x'(t))^2}{(x'(t))^2 + 1} \times \frac{1}{x(t)\left(1 + x^2(t)\right)^2} \\ &\le \sin t \\ &= p(t), \quad \forall x', x \in \mathbb{R} \text{ and } t \ge t_0 = 1, \end{aligned}$$

and

$$\int_{\pm\varepsilon}^{\pm\infty} \frac{\psi(y)}{g(y)} dy = \int_{\pm\varepsilon}^{\pm\infty} \frac{y}{(1 + y^2)^2} dy = \frac{1}{2\left(1 + \varepsilon^2\right)} < \infty.$$

If $r(t)=\frac{1}{t}$ and $\rho(t)=1$ then, $\left(\rho(t)r(t)\right)'=-\frac{1}{t^2}<0$ and $\int_{t_0}^{\infty}\rho(s)ds=\int_{1}^{\infty}ds=\infty.$

Therefore,

$$\limsup_{t\to\infty}\left[\int_{t_0}^{t}\rho(s)ds\right]^{-1}\int_{t_0}^{t}\rho(s)\left\{\int_{t_0}^{s}\left[q(u)-p(u)\right]du\right\}ds$$
$$=\limsup_{t\to\infty}\left[\int_{1}^{t}ds\right]^{-1}\int_{1}^{t}\left\{\int_{1}^{s}\left[u^3-\sin u\right]du\right\}ds$$
$$=\limsup_{t\to\infty}\frac{1}{t-1}\left[\frac{t^5}{20}+\sin t-\frac{1}{4}t-t\cos 1-\frac{1}{20}+\sin 1-\frac{1}{4}-\cos 1\right]$$
$$=\infty.$$

Hence, this solution is oscillatory by Theorem 2.2.

Remark 2.2. If $r(t)=1,\ \psi(x(t))=1,\ f(x'(t))=x',\ g(x(t))=x$ and $H(t,x(t),x'(t))=0,$ then Theorem 2.2 is reduced to the theorem as in Wintner (1949).

Remark 2.3. Remili (2008) has formed a number of oscillation conditions results for Eq. (1) with

$$\psi(x)=1 \text{ and } f(x'(t))=x'.$$

These results require that

$$a(t)\le a_1,$$

and

$$\liminf_{t\to\infty}\int_{T}^{t}Z(s)ds>-\lambda\,(\lambda>0) \text{ for all large } T;\ Z(s)=R(s)\left[q(s)-p(s)\right].$$

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