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NEW SUBCLASSES OF ANALYTIC AND *M*-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS DEFINED BY DIFFERENTIAL OPERATOR

(Subkelas Baharu Fungsi Analisis dan Bi-Univalen Simetri *m*-Lipat yang Ditakrif oleh Pengoperasi Pembeza**)**

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ABSTRACT

By using generalized differential operator, we introduce two new subclasses of analytic and *m*-fold symmetric bi-univalent functions in the open unit disk U. We also find coefficient estimates of $|a_2|$ and $|a_3|$ for these new subclasses.

Keywords: Differential operator; *m*-fold symmetry; bi-univalent functions; coefficient bounds

ABSTRAK

Dengan menggunakan pengoperasi pembeza, diperkenalkan dua subkelas baharu fungsi analisis dan bi-univalen simetri *m*-lipat dalam cakera unit terbuka U. Anggaran pekali bagi $|a_2|$ dan $|a_3|$ untuk subkelas fungsi baharu ini turut diperoleh.

Kata kunci: Pengoperasi pembeza; simetri *m*-lipat; fungsi bi-univalen; batas pekali

1. Introduction

Let A denote a class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0.$

We denote by S the class of all functions in A which are univalent in U. The Koebe one-quarter Theorem (Duren 1983) states that the image of U under every function f from S contains a disk of radius $1/4$. Thus every function $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z, \qquad (z \in U), \tag{2}$$

and

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f), \; r_0(f) \geq \frac{1}{4}. \tag{3}$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w)=w-a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^2-5a_2a_3+a_4\right)w^4+\ldots. \tag{4}$$

A function $f\in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let $\sum$ denote the class of bi-univalent functions in U given by (1).

Some examples for bi-univalent functions are given as:

$$h_1(z)=\frac{z}{1-z},\ h_2(z)=-\log(1-z),\ h_3(z)=\frac{1}{2}\log\left(\frac{1+z}{1-z}\right),\ z\in U.$$

Lewin (1967) investigated the class $\sum$ of bi-univalent functions and obtained a bound given by $|a_2|\le 1.5$. Motivated by (Brannan & Clunie 1980; Lewin 1967), conjectured that $|a_2|\le\sqrt{2}$. For more work on bi-univalent one can refer to Altinkaya and Yalcin (2015); Aouf *et al.* (2013); Feili and Wang (2012); Hussain *et al.* (2018); Jothibasu (2015); Porwal and Darus (2013); Srivastava *et al.* (2013).

For each function $f\in S$, the function

$$h(z)=\sqrt[m]{f\left(z^m\right)}\quad\left(z\in U,\ m\in N\right) \tag{5}$$

is univalent and maps the unit disk U into a region with *m*-fold symmetry. We denote by S_m the class of *m*-fold symmetric univalent functions in U and clearly $S_1=S$. If $f\in S_m$ then it has a series expansion of the form

$$f(z)=z+\sum_{l=1}^{\infty}a_{ml+1}z^{ml+1}\quad. \tag{6}$$

In recent years, the study of bi-univalent functions has gained a lot of attention due to the work of Srivastava *et al.* (2010). Also Srivastava *et al.* (2013), introduced a natural extensions of m-fold symmetric univalent functions and defined the class $\sum_m$ of symmetric bi-univalent functions. They obtained the series expansion for $g=f^{-1}$ as:

$$\begin{aligned}g(w)=f^{-1}(w)=w-a_{m+1}w^{m+1}+\left[(m+1)a_{m+1}^2-a_{2m+1}\right]w^{2m+1}-\\ \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3-(3m+2)a_{m+1}a_{2m+1}+a_{3m+1}\right]w^{3m+1}+\ldots\end{aligned} \tag{7}$$

where f is given by (6). For $m=1$, the formula (7) coincides with the formula (4) of the class $\sum$.

We introduce the class $Q(\varepsilon,\mu)$ of analytic functions defined as follows:

$$Q(\mu,\varepsilon)=\left\{ f\in A:\operatorname{Re}\left(\frac{z\left[1+zf''(z)\right]}{\mu f(z)+z(1-\mu)f'(z)}\right)>\varepsilon, 0\le\varepsilon<1,\mu\ge 0\right\}. \tag{8}$$

It is easy to see that $Q(\mu_1,\varepsilon)\subset Q(\mu_2,\varepsilon)$ for $\mu_1>\mu_2\ge 0$. Thus for $\mu\ge 1,\ 0\le\varepsilon<1, Q(\mu,\varepsilon)\subset Q(\mu_1,\varepsilon)=\left\{f\in A;\operatorname{Re}f'(z)>\varepsilon,\ 0\le\varepsilon<1\right\}$ and hence $Q(\mu,\varepsilon)$ is univalent, see Chichra (1976).

Now, $D^k_{\alpha,\beta,\lambda,\delta}$ is the operator introduced by Ramadan and Darus (2010), such that

$$D^0 f(z)=f(z),$$

$$D^1_{\alpha,\beta,\lambda,\delta} f(z)=\left[1-(\lambda-\delta)(\beta-\alpha)\right]f(z)+(\lambda-\delta)(\beta-\alpha)zf'(z)$$

$$=z+\sum_{n=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]a_n z^n$$

$$\vdots$$

$$D^k_{\alpha,\beta,\lambda,\delta} f(z)=z+\sum_{n=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k a_n z^n$$

for $\alpha\ge 0,\beta\ge 0,\lambda\ge 0,\delta\ge 0,\lambda>\delta,\beta>\alpha$, $k\in\{0,1,2,...\}$.

When $\alpha=0,\ \delta=0,\ \beta=1,\ \lambda=1$ it reduces to Sãlãgean (1983) differential operator. It reduces to Darus and Ibrahim (2009), when $\alpha=0$. It reduces to Al-Oboudi (2004) differential operator when $\alpha=0,\ \delta=0,\ \beta=1$.

In order to establish our main results, we need the following lemma.

Lemma 1.1. If $p(z)=1+p_1 z+p_2 z^2+p_3 z^3+...$ is an analytic function in U for which $\operatorname{Re}\{p(z)\}>0$, then $|p_n|\le 2,\quad (n\in N=\{1,2,3,...\})$.

1.1. *The class* $S_{\sum_m}(\mu,\gamma,\alpha,\beta,\lambda,\delta)$

Definition 1.2. A function $f\in\sum_m$ is said to be in the class $S_{\sum_m}(\mu,\gamma,\alpha,\beta,\lambda,\delta)$ for $(0<\gamma\le 1,\ 0\le\mu<1, z\in U)$ if the following conditions are satisfied:

$$\left|\arg\left\{\frac{z\left[1+z\left(D\left(D_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)\right)\right)\right]}{\mu D_{\alpha,\beta,\lambda,\delta}^{k}f(z)+z(1-\mu)D_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}\right\}\right|<\frac{\gamma\pi}{2},$$

and

$$\left|\arg\left\{\frac{z\left[1+z\left(D\left(D_{\alpha,\beta,\lambda,\delta}^{k+1}g(z)\right)\right)\right]}{\mu D_{\alpha,\beta,\lambda,\delta}^{k}g(z)+z(1-\mu)D_{\alpha,\beta,\lambda,\delta}^{k+1}g(z)}\right\}\right|<\frac{\gamma\pi}{2}.$$

We note that for $k=0, m=1$, we obtain a new class of bi-univalent function

$S_{\sum_m}(\mu,\gamma,\alpha,\beta,\lambda,\delta)=S_{\sum}(\mu,\gamma)$. For k = 0, we obtain a new class which consists m-fold symmetric bi-univalent function $S_{\sum_m}(\mu,\gamma,\alpha,\beta,\lambda,\delta)=S_{\sum_m}(\mu,\gamma)$.

1.2. *The class* $S_{\sum_m}(\mu,\eta,\alpha,\beta,\lambda,\delta)$

Definition 1.3. A function $f\in\sum_m$ is said to be in the class $S_{\sum_m}(\mu,\eta,\alpha,\beta,\lambda,\delta)$ for $(0<\gamma\le 1, 0\le\mu<1, z\in U)$ if the following conditions are satisfied

$$\left(\frac{z\left[1+z\left(D\left(D_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)\right)\right)\right]}{\mu D_{\alpha,\beta,\lambda,\delta}^{k}f(z)+z(1-\mu)D_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}\right)>\eta,$$

$$\left(\frac{z\left[1+z\left(D\left(D_{\alpha,\beta,\lambda,\delta}^{k+1}g(z)\right)\right)\right]}{\mu D_{\alpha,\beta,\lambda,\delta}^{k}g(z)+z(1-\mu)D_{\alpha,\beta,\lambda,\delta}^{k+1}g(z)}\right)>\eta.$$

We note that for $k=0, m=1$, we obtain a new class of bi-univalent function

$S_{\sum_m}(\mu,\eta,\alpha,\beta,\lambda,\delta)=S_{\sum_m}(\mu,\eta)$. For $k=0$, we obtain a new class which consists m-fold symmetric bi-univalent function $S_{\sum_m}(\mu,\eta,\alpha,\beta,\lambda,\delta)=S_{\sum_m}(\mu,\eta)$.

2. Main results

Theorem 2.1. *Let* $f \in \sum_m$ *be in the class* $S_{\sum_m}(\mu,\gamma,\alpha,\beta,\lambda,\delta)$ *then,*

$$|a_{m+1}| \le \frac{2\gamma}{\sqrt{\left\{\gamma\left(4m^2+2m\mu-1\right)(m+1)+\psi\left[\gamma(m\mu-m-1)-\psi(\gamma-1)\right]\right\}\varphi_1}},$$

$$|a_{2m+1}| \le \frac{2\gamma}{\left(4m^2+2m\mu-1\right)\varphi_2}+$$

$$\frac{4(m+1)}{\varphi_2}\left[\frac{\gamma}{\gamma\left(4m^2+2m\mu-1\right)(m+1)+\psi\left[\gamma(m\mu-m-1)-\psi(\gamma-1)\right]}\right],$$

where $\varphi_n = \left[mn(\lambda-\delta)(\beta-\alpha)+1\right]^k$, $\psi = \left(m^2+\mu m-1\right)$

Proof. Let $f \in S_{\sum_m}(\mu,\gamma,\alpha,\beta,\lambda,\delta)$, then

$$\frac{z\left[1+z\left(D\left(D^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)\right)\right)\right]}{\mu D^{k}_{\alpha,\beta,\lambda,\delta}f(z)+z(1-\mu)D^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)} = \left[p(z)\right]^{\gamma}, \tag{9}$$

and

$$\frac{w\left[1+w\left(D\left(D^{k+1}_{\alpha,\beta,\lambda,\delta}g(w)\right)\right)\right]}{\mu D^{k}_{\alpha,\beta,\lambda,\delta}g(w)+w(1-\mu)D^{k+1}_{\alpha,\beta,\lambda,\delta}g(w)} = \left[q(w)\right]^{\gamma}. \tag{10}$$

where $g = f^{-1}$ and $p,\ q$ in P have the following forms:

$$\begin{aligned} p(z) &= 1+p_m z^m + p_{2m}z^{2m}+\ldots \\ q(z) &= 1+q_m w^m + q_{2m}w^{2m}+\ldots. \end{aligned} \tag{11}$$

Now, equating the coefficients in (9), we get

$$\left(m^2+m\mu-1\right)a_{m+1}\varphi_1 = \gamma p_m, \tag{12}$$

$$\begin{aligned} &\left(4m^2+2\mu m-1\right)a_{2m+1}\varphi_2 - \left[\left(m^2+\mu m-1\right)(m-\mu m+1)\right]\varphi_1^2 a_{m+1}^2 \\ &= \gamma p_{2m} + \frac{\gamma(\gamma-1)}{2}p_m^2. \end{aligned} \tag{13}$$

$$-\left(m^2+m\mu-1\right)a_{m+1}\varphi_1 = \gamma q_m \tag{14}$$

$$\left[\left(4m^2+2\mu m-1\right)\left(m+1\right)+\left(m^2+\mu m-1\right)\left(\mu m-m-1\right)\right]\varphi_1^2 a_{m+1}^2$$
$$-\left(4m^2+2\mu m-1\right)a_{2m+1}\varphi_2 = \gamma q_{2m}+\frac{\gamma\left(\gamma-1\right)}{2}q_m^2. \qquad (15)$$

From (12) and (14), we obtain

$$p_m = -q_m, \qquad (16)$$

$$2\left(m^2+m\mu-1\right)^2 a_{m+1}^2\varphi_1^2 = \gamma^2\left(p_m^2+q_m^2\right) \qquad (17)$$

Also from (13), (15) and (17), we have

$$\left\{\left(4m^2+2\mu m-1\right)\left(m+1\right)+2\left(m^2+\mu m-1\right)\left(\mu m-m-1\right)\right\}\varphi_1^2 a_{m+1}^2 =$$
$$\gamma\left(p_{2m}+q_{2m}\right)+\frac{\gamma\left(\gamma-1\right)}{2}\left(p_m^2+q_m^2\right). \qquad (18)$$

$$=\gamma\left(p_{2m}+q_{2m}\right)$$
$$+\frac{\gamma\left(\gamma-1\right)}{2}\frac{2\left(m^2+m\mu-1\right)^2\varphi_1^2}{\gamma^2}a_{m+1}^2.$$

Therefore, we have

$$a_{m+1}^2=\frac{\gamma^2\left(p_{2m}+q_{2m}\right)}{\left\{\gamma\left(4m^2+2m\mu-1\right)\left(m+1\right)+\psi\left[2\gamma\left(m^2-\mu m-1\right)-\psi\left(\gamma-1\right)\right]\right\}\varphi_1^2}$$

Applying Lemma 1.1., for coefficients p_{2m} and q_{2m}, we obtain

$$\left|a_{m+1}\right|\le\frac{2\gamma}{\sqrt{\left\{\gamma\left(4m^2+2m\mu-1\right)\left(m+1\right)+\psi\left[2\gamma\left(m^2-\mu m-1\right)-\psi\left(\gamma-1\right)\right]\right\}}\,\varphi_1}\ . \qquad (19)$$

Next, in order to find the bound on $\left|a_{2m+1}\right|$, by subtracting (15) from (13), we obtain

$$2\left(4m^2+2\mu m-1\right)a_{2m+1}\varphi_2-\left(4m^2+2m\mu-1\right)\varphi_1^2 a_{m+1}^2$$
$$=\gamma\left(p_{2m}-q_{2m}\right)+\frac{\gamma\left(\gamma-1\right)}{2}\left(p_m^2-q_m^2\right) \qquad (20)$$

Then, in view of (16) and (17) and applying Lemma 1.1 for coefficients p_{2m}, p_m and q_{2m}, q_m, we have

$$|a_{2m+1}|\leq\frac{2\gamma}{\left(4m^2+2m\mu-1\right)\varphi_2}+$$
$$\frac{4(m+1)}{\varphi_2}\left[\frac{\gamma}{\gamma\left(4m^2+2m\mu-1\right)(m+1)+\psi\left[\gamma(m\mu-m-1)-\psi(\gamma-1)\right]}\right]. \tag{21}$$

This completes the proof of Theorem 2.1.

For $k=0$, in Theorem 2.1, we have the following corollary.

Corollary 2.2. Let f given by (6) be in the class $f\in S_{\sum_m}(\mu,\gamma)$, then

$$|a_{m+1}|\leq\frac{2\gamma}{\sqrt{\gamma\left(4m^2+2m\mu-1\right)(m+1)+\psi\left[2\gamma\left(m^2-\mu m-1\right)-\psi(\gamma-1)\right]}}$$

and

$$|a_{2m+1}|\leq\frac{2\gamma}{\left(4m^2+2m\mu-1\right)}+$$
$$\left[\frac{4(m+1)\gamma}{\gamma\left(4m^2+2m\mu-1\right)(m+1)+\psi\left[\gamma(m\mu-m-1)-\psi(\gamma-1)\right]}\right].$$

For $m=1$, in Theorem 2.1, we have the following corollary.

Corollary 2.3. Let f given by (6) be in the class $f\in S_{\sum}(\mu,\gamma,\alpha,\beta,\lambda,\delta)$, then

$$|a_2|\leq\frac{2\gamma}{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k\sqrt{\left[\gamma\left(2\mu^2+6\right)-\mu(\gamma-1)\right]}}$$

and

$$|a_3|\leq\frac{2\gamma}{\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^k}\left\{\frac{2\gamma\mu^2-\mu\gamma+9\mu+6\gamma+12}{(3+2\mu)\left[\gamma\left(2\mu^2+6\right)-\mu(\gamma-1)\right]}\right\}.$$

For $k=0$, $m=1$, in Theorem 2.1, we have the following corollary.

Corollary 2.4. Let f given by (6) be in the class $f \in S_{\sum}(\mu,\gamma)$, then

$$|a_2| \le \frac{2\gamma}{\sqrt{\left[\gamma\left(2\mu^2+6\right)-\mu(\gamma-1)\right]}}$$

and

$$|a_3| \le \frac{\gamma\left(2\mu^2-\mu+6\gamma\right)+9\mu+12}{\mu(3+2\mu)+\gamma\left(4\mu^3+4\mu^2+9\mu+18\right)}.$$

Theorem 2.5. *let* $f \in \sum_m$ *be in the class* $S_{\sum_m}(\mu,\eta,\alpha,\beta,\lambda,\delta)$ *then,*

$$|a_{m+1}| \le \frac{\sqrt{2(1-\eta)}}{\sqrt{\left(4m^2+2\mu m-1\right)(m+1)\varphi_1^2}},$$

$$|a_{2m+1}| \le \frac{(1-\eta)(m+2)}{\left(4m^2+2m\mu-1\right)(m+1)\varphi_2}+\frac{2\left(m^2+m\mu-1\right)(m-m\mu+1)}{\left(4m^2+2m\mu-1\right)\varphi_2},$$

where $\varphi_n = \left[mn(\lambda-\delta)(\beta-\alpha)+1\right]^k$

Proof. Let $f \in S_{\sum_m}(\eta,\mu,\alpha,\beta,\lambda,\delta)$, then

$$\frac{z\left[1+z\left(D\left(D^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)\right)\right)\right]}{\mu D^{k}_{\alpha,\beta,\lambda,\delta}f(z)+z(1-\mu)D^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)} = \eta+(1-\eta)p(z), \tag{22}$$

and for its inverse map $g = f^{-1}$, we have

$$\frac{z\left[1+z\left(D\left(D^{k+1}_{\alpha,\beta,\lambda,\delta}g(z)\right)\right)\right]}{\mu D^{k}_{\alpha,\beta,\lambda,\delta}g(z)+z(1-\mu)D^{k+1}_{\alpha,\beta,\lambda,\delta}g(z)} = \eta+(1-\eta)q(z), \tag{23}$$

where $p(z)$ and $q(z)$ have the forms (3). Equating coefficients in (22) and (23) yields

$$\left(m^2+m\mu-1\right)a_{m+1}\varphi_1 = (1-\eta)p_m, \tag{24}$$

$$\begin{aligned}&\left(4m^2+2\mu m-1\right)a_{2m+1}\varphi_2-\left[\left(m^2+\mu m-1\right)(m-\mu m+1)\right]\varphi_1^2a_{m+1}^2\\&=(1-\eta)p_{2m},\end{aligned} \tag{25}$$

$$-\left(m^2+m\mu-1\right)a_{m+1}\varphi_1=(1-\eta)q_m\ , \tag{26}$$

$$\left[\left(4m^2+2\mu m-1\right)(m+1)-\left(m^2+\mu m-1\right)(\mu m-m-1)\right]\varphi_1^2a_{m+1}^2$$
$$-\left(4m^2+2\mu m-1\right)a_{2m+1}\varphi_2=(1-\eta)q_{2m} \tag{27}$$

From (24) and (26), we have

$$p_1=-q_1 \tag{28}$$

and

$$2\left(m^2+m\mu-1\right)^2a_{m+1}^2\varphi_1^2=(1-\eta)^2\left(p_m^2+q_m^2\right) \tag{29}$$

Also, from (25) and (27), we find that

$$\left(4m^2+2\mu m-1\right)(m+1)\varphi_1^2a_{m+1}^2=(1-\eta)\left(p_{2m}+q_{2m}\right) \tag{30}$$

or

$$a_{m+1}^2=\frac{(1-\eta)\left(p_{2m}+q_{2m}\right)}{\left(4m^2+2\mu m-1\right)(m+1)\varphi_1^2}$$

$$\left|a_{m+1}\right|\le\frac{\sqrt{2(1-\eta)}}{\sqrt{\left(4m^2+2\mu m-1\right)(m+1)\varphi_1^2}}$$

Next, in order to find the bound on $\left|a_{2m+1}\right|$, by subtracting (27) from (25), we obtain

$$2\left(4m^2+2\mu m-1\right)a_{2m+1}\varphi_2-\left\{2\left(m^2+\mu m-1\right)(m-\mu m+1)\right.$$
$$\left.+\left(4m^2+2m\mu-1\right)\right\}\varphi_1^2a_{m+1}^2=(1-\eta)\left(p_{2m}-q_{2m}\right)$$

Then, in view of (28) and (29) and applying Lemma 1.1 for coefficients p_{2m}, p_m and q_{2m}, q_m, we have

$$\left|a_{2m+1}\right|\le\frac{(1-\eta)(m+2)}{\left(4m^2+2m\mu-1\right)(m+1)\varphi_2}+\frac{2\left(m^2+m\mu-1\right)(m-m\mu+1)}{\left(4m^2+2m\mu-1\right)\varphi_2}\quad ,$$

which completes the proof of theorem

For $k=0$, in Theorem 2.5, we have the following corollary.

Corollary 2.6. Let f given by (6) be in the class $S_{\sum_m}(\mu,\eta)$, then

$$|a_{m+1}| \leq \frac{\sqrt{2(1-\eta)}}{\sqrt{(4m^2+2\mu m-1)(m+1)}}$$

and

$$|a_{2m+1}| \leq \frac{(1-\eta)(m+2)}{(4m^2+2m\mu-1)(m+1)} + \frac{2(m^2+m\mu-1)(m-m\mu+1)}{(4m^2+2m\mu-1)}$$

For $m=1$, in Theorem 2.5, we have the following corollary.

Corollary 2.7. Let f given by (6) be in the class $S_{\sum}(\mu,\eta,\alpha,\beta,\lambda,\delta)$, then

$$|a_2| \leq \frac{1}{[(\lambda-\delta)(\beta-\alpha)+1]^k}\sqrt{\frac{(1-\eta)}{(3+2\mu)}}$$

and

$$|a_3| \leq \frac{3(1-\eta)+4\mu^2}{2(3+2\mu)[2(\lambda-\delta)(\beta-\alpha)+1]^k}$$

For $k=0$, $m=1$ in Theorem 2.5., we have the following corollary.

Corollary 2.8. Let f given by (6) be in the class $S_{\sum}(\mu,\eta)$, then

$$|a_2| \leq \sqrt{\frac{(1-\eta)}{(3+2\mu)}},$$

and

$$|a_3| \leq \frac{3(1-\eta)+4\mu^2}{2(3+2\mu)}.$$

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