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FOURTH HANKEL DETERMINANT FOR A SUBCLASS OF STARLIKE FUNCTIONS

(Penentu Hankel Keempat untuk Suatu Subkelas Fungsi Bak Bintang)

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ABSTRACT

Let B be the class of normalized starlike functions $f(z)$ which are analytic on an open unit disc $E = \{z : |z| < 1\}$ and satisfy

$$\text{Re}\{\alpha f'(z) + \beta z f''(z)\} > 0$$

for some $0 < \alpha \le 1$ and $0 \le \beta < 1$. In this paper, we will obtain a sharp bound of the fourth Hankel determinant, $H_4(1)$.

Keywords: analytic function; Hankel determinant; starlike function

ABSTRAK

Lambangkan B sebagai kelas fungsi, $f(z)$ yang terdiri daripada fungsi analisis di dalam cakera unit terbuka $E = \{z : |z| < 1\}$ dan memenuhi syarat

$$\text{Re}\{\alpha f'(z) + \beta z f''(z)\} > 0$$

untuk $0 < \alpha \le 1$, $0 \le \beta < 1$ dan $f(z)$ juga adalah fungsi bak bintang ternormal. Makalah ini bertujuan untuk mendapatkan batas atas terbaik bagi penentu Hankel keempat, $H_4(1)$.

Keywords: fungsi analisis; penentu Hankel, fungsi bak bintang

1. Introduction

Let A be the family of all analytic functions f that having the Taylor series expansions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. A function $f \in A$ is said to be the class of bounded turning, R and class of starlike, S^* if and only if it satisfies

$$\text{Re}\{f'(z)\} > 0 \text{ and } \text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$$

respectively.

Let P denote the class of functions consisting of p, such that

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{1.2}$$

which are regular in the open unit disc E and satisfy $\operatorname{Re} p(z) > 0$ for any $z \in E$, where $p(z)$ is called the Caratheodory function. It is well known that the *nth* coefficient is bounded by *n*. The bounds for the coefficients give information about the geometric properties of these functions.

The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke (2010) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.3}$$

The study on second Hankel determinant, $H_2(2)$ was started since 1960 (Yahya *et al.*, 2013). Third Hankel determinant was first obtained by Babalola (2007) for the families of R and S^* defined as

$$H_3(1) = a_3\left(a_2a_3 - a_3^2\right) - a_4\left(a_2a_3 - a_4\right) + a_5\left(a_3 - a_2^2\right) \tag{1.4}$$

which is more complex. The results of this determinant also found by other researchers such as Krishna *et al.* (2015), Mohamed *et al.* (2018) and Mohamed *et al.* (2019) for some generalized subclasses of analytic function. For our discussion, we consider the fourth Hankel determinant,

$$H_4(1) = a_7H_3(1) + a_6\Delta_1 + a_5\Delta_2 + a_4\Delta_3 \tag{1.5}$$

where

$$\begin{aligned} \Delta_1 &= \left(a_3a_6 - a_4a_6\right) - a_2\left(a_2a_6 - a_3a_5\right) + a_4\left(a_2a_4 - a_3^2\right), \\ \Delta_2 &= \left(a_4a_6 - a_5^2\right) - a_5\left(a_3a_6 - a_4a_5\right) + a_3\left(a_3a_5 - a_4^2\right), \\ \Delta_3 &= a_2\left(a_4a_6 - a_5^2\right) - a_3\left(a_3a_6 - a_4a_5\right) + a_4\left(a_3a_5 - a_4^2\right). \end{aligned} \tag{1.6}$$

Motivated from Arif *et al.* (2018) that obtained $H_4(1)$ for the class R, we will find the upper bound of the functional $H_4(1)$ for the class of function defined by Mohamed *et al.* (2012a).

Definition 1.1. Let B denoted the class of functions $f(z)$ which are analytic with an open unit disc E and satisfy

$$\operatorname{Re}\left\{\alpha f'(z) + \beta z f''(z)\right\} > 0$$

for some $0 < \alpha \leq 1$ and $0 \leq \beta < 1$.

2. A Set of Lemmas

In order to find the bound of $H_4(1)$, we need the following sharp estimations.

Lemma 2.1. (Pommerenke 2010) *If* $p \in P$, *then* $|c_n| \le 2$ *for each* $n = 1, 2, 3, \ldots$ *and the inequality is sharp for the function* $\dfrac{1+z}{1-z}$.

Lemma 2.2 (Libera & Zlotkiewicz 1982; 1983) *Let* $p \in P$,

$$2c_2 = c_1^2 + \left(4 - c_1^2\right)x \tag{2.1}$$

for some x, $|x| \le 1$ *and*

$$4c_3 = c_1^3 + 2c_1\left(4 - c_1^2\right)x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z \tag{2.2}$$

for some z, $|z| \le 1$.

Lemma 2.3 (Mohamed *et al.* 2012a) *If* $f(z) \in B$, *then*

$$\alpha f'(z) + \beta z f''(z) = \alpha p(z) \tag{2.3}$$

where $0 < \alpha \le 1$ *and* $0 \le \beta < 1$.

Lemma 2.4 (Mohamed *et al.* 2012a) *If* $f(z) \in B$, *then*

$$|a_n| \le \frac{2\alpha}{n\left(\alpha + (n-1)\beta\right)}, n \in \{1, 2, 3, \ldots\} \tag{2.4}$$

where $0 < \alpha \le 1$ *and* $0 \le \beta < 1$.

Lemma 2.5 (Mohamed *et al.* 2012b) *If* $f(z) \in B$, *then*

$$\left|a_2 a_3 - a_3^2\right| \le \frac{4\alpha^2}{9(\alpha + 2\beta)^2} \tag{2.5}$$

where $0 < \alpha \le 1$ *and* $0 \le \beta < 1$. *The result obtained is sharp.*

3. Main Results

Our main results are focusing on finding the bound for $\left|a_2a_3-a_4\right|$, $\left|a_3-a_2^2\right|$, $\left|a_n\right|$, $\left|\Delta_1\right|$, $\left|\Delta_2\right|$ and $\left|\Delta_3\right|$ to get the bound of $H_4(1)$.

Theorem 3.1. *For* $f\in B$, *then*

$$\left|a_2a_3-a_4\right|\le\frac{\alpha\left(3T-2(\alpha+3\beta)\alpha\right)}{6T(\alpha+3\beta)}\left[\frac{9T-4(\alpha+3\beta)\alpha}{3\left(3T-2(\alpha+3\beta)\alpha\right)}\right]^{\frac{3}{2}} \tag{3.1}$$

where $0<\alpha\le1, 0\le\beta<1$ *and* $T=(\alpha+\beta)(\alpha+2\beta)$.

Proof. Let $f\in B$. Using Lemma 2.3 and with some simplifications, we can obtain

$$a_2a_3-a_4=\frac{\alpha\left(2c_1c_2(\alpha+3\beta)-3c_3(\alpha+\beta)(\alpha+2\beta)\right)}{12(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)}. \tag{3.2}$$

Applying triangle inequalities for (3.2), we get

$$\left|a_2a_3-a_4\right|=\left|\frac{\alpha}{12(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)}\right|\left|\left(2c_1c_2(\alpha+3\beta)-3c_3(\alpha+\beta)(\alpha+2\beta)\right)\right|. \tag{3.3}$$

Consider the right-hand side of (3.3) and used Lemma 2.2 to obtain

$$\begin{aligned}&4\left|2c_1c_2(\alpha+3\beta)-c_3(\alpha+\beta)(\alpha+2\beta)\right|\\ &\le\left|\begin{aligned}&4\alpha c_1(\alpha+3\beta)\left(c_1^2+\left(4-c_1^2\right)\right)x-3(\alpha+\beta)(\alpha+2\beta)\left(c_1^3+2c_1\left(4-c_1^2\right)\right)x\\ &-c_1\left(4-c_1^2\right)+2\left(4-c_1^2\right)\left(1-|x|^2z\right)\end{aligned}\right|\end{aligned} \tag{3.4}$$

and again, by the application of triangle inequalities for (3.4) and let $c_1=c$,

$$\begin{aligned}&4\left|2c_1c_2(\alpha+3\beta)-3c_3(\alpha+\beta)(\alpha+2\beta)\right|\\ &=\left(4\alpha(\alpha+3\beta)-3(\alpha+\beta)(\alpha+2\beta)\right)c^3+3(\alpha+\beta)(\alpha+2\beta)\left(4-c^2\right)c\phi^2\\ &\quad-\left(4\alpha(\alpha+3\beta)\left(4-c^2\right)\right)c\phi+6(\alpha+\beta)(\alpha+2\beta)\left(4-c^2\right)c\phi\\ &\quad+6(\alpha+\beta)(\alpha+2\beta)\left(4-c^2\right)-6(\alpha+\beta)(\alpha+2\beta)\left(4-c^2\right)\phi^2\\ &=F(c,\phi).\end{aligned}$$

Differentiate $F(c,\phi)$ with respect to ϕ we obtain

$$\frac{\partial F}{\partial \phi}=6(\alpha+\beta)(\alpha+2\beta)\left(4-c^2\right)c\phi+6(\alpha+\beta)(\alpha+2\beta)\left(4-c^2\right)c$$
$$-12(\alpha+\beta)(\alpha+2\beta)\left(4-c^2\right)\phi-4\alpha(\alpha+3\beta)\left(4-c^2\right)c$$

for $\phi\in(0,1)$, fixed $c\in(0,2)$, $\alpha\in(0,1]$ and $\beta\in[0,1)$. Then, we observe $\frac{\partial F}{\partial \phi}>0.$

Therefore, $F(c,\phi)$ becomes an increasing function of ϕ and consequently it cannot obtain maximum value on the closed region $[0,2]\times[0,1]$. For fixed $c\in[0,2]$, we have

$$\max_{0\le\phi\le1}F(c,\phi)=F(c,1)=G(c).$$

By simplifying the relation, we can obtain

$$G(c)=8\alpha(\alpha+3\beta)c^3+36(\alpha+\beta)(\alpha+2\beta)c-16\alpha(\alpha+3\beta)c$$
$$-12c^3(\alpha+\beta)(\alpha+2\beta), \tag{3.5}$$

therefore

$$G'(c)=-36c^2(\alpha+\beta)(\alpha+2\beta)+24c^2(\alpha+3\beta)\alpha$$
$$+36(\alpha+\beta)(\alpha+2\beta)-16(\alpha+3\beta)\alpha. \tag{3.6}$$

and

$$G''(c)=-72c(\alpha+\beta)(\alpha+2\beta)+48c(\alpha+3\beta)\alpha. \tag{3.7}$$

In order to get optimum value of c, we consider $G'(c)=0$ from (3.6)

$$c=\sqrt{\frac{9(\alpha+\beta)(\alpha+2\beta)-4(\alpha+3\beta)\alpha}{9(\alpha+\beta)(\alpha+2\beta)-6(\alpha+3\beta)\alpha}}. \tag{3.8}$$

Substituting the value c from (3.8) into (3.7)

$$G''(c)=-24\big(3(\alpha+\beta)(\alpha+2\beta)\big)\left(\sqrt{\frac{9(\alpha+\beta)(\alpha+2\beta)-4(\alpha+3\beta)\alpha}{3\big(3(\alpha+\beta)(\alpha+2\beta)-2(\alpha+3\beta)\alpha\big)}}\right)$$
$$+48\left(\sqrt{\frac{9(\alpha+\beta)(\alpha+2\beta)-4(\alpha+3\beta)\alpha}{3\big(3(\alpha+\beta)(\alpha+2\beta)-2(\alpha+3\beta)\alpha\big)}}\right)(\alpha+3\beta)\alpha<0. \tag{3.9}$$

Therefore, the second derivative of $G(c)$ has maximum value at c, where c is (3.8). By using the obtained value in (3.8), which simplifies to give the maximum value of (3.6) as

$$G_{\max} = 8\left(3(\alpha+\beta)(\alpha+2\beta)-2(\alpha+3\beta)\alpha\right)\left(\frac{9(\alpha+\beta)(\alpha+2\beta)-4(\alpha+3\beta)\alpha}{3\left(3(\alpha+\beta)(\alpha+2\beta)-2(\alpha+3\beta)\alpha\right)}\right)^{\frac{3}{2}}. \tag{3.10}$$

Simplify the equation of (3.10), we get

$$\begin{aligned}&\left|2c_1c_2(\alpha+3\beta)-3c_3(\alpha+\beta)(\alpha+2\beta)\right| \\ &\leq 2\left(3(\alpha+\beta)(\alpha+2\beta)-2(\alpha+3\beta)\alpha\right)\left(\frac{9(\alpha+\beta)(\alpha+2\beta)-4(\alpha+3\beta)\alpha}{3\left(3(\alpha+\beta)(\alpha+2\beta)-2(\alpha+3\beta)\alpha\right)}\right)^{\frac{3}{2}}.\end{aligned} \tag{3.11}$$

Simplification of the above relation and (3.3), we obtain

$$|a_2a_3-a_4| \leq \frac{\alpha\left(3(\alpha+\beta)(\alpha+2\beta)-2(\alpha+3\beta)\alpha\right)}{6(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)}\left[\frac{9(\alpha+\beta)(\alpha+2\beta)-4(\alpha+3\beta)\alpha}{3\left(3(\alpha+\beta)(\alpha+2\beta)-2(\alpha+3\beta)\alpha\right)}\right]^{\frac{3}{2}}.$$

This complete the proof of Theorem 3.1. □

Theorem 3.2. *For* $f \in B$*, then*

$$|a_3-a_2^2| \leq \frac{2}{3}\frac{(\alpha+\beta)}{(\alpha+2\beta)^2}. \tag{3.12}$$

where $0<\alpha\leq 1$ *and* $0\leq\beta<1$.

Proof. Let $f \in B$. Also, using Lemma 2.3, we can have a_3 and a_2 to substitute in $|a_3-a_2^2|$ which give us

$$|a_3-a_2^2| = \left|\frac{1}{12}\right|\left|\frac{\alpha\left(4c_2(\alpha+\beta)^2-3\alpha c_1^2(\alpha+2\beta)\right)}{(\alpha+2\beta)(\alpha+\beta)^2}\right|. \tag{3.13}$$

Suppose that $c_1 = c$ where $c \in [0,2]$,

$$|a_3-a_2^2| = \left|\frac{1}{12}\right|\left|\frac{\alpha\left(2\left(c(\alpha+\beta)^2+x\left(4(\alpha+\beta)^2-c^2(\alpha+\beta)^2\right)\right)-3\alpha c^2(\alpha+2\beta)\right)}{(\alpha+2\beta)(\alpha+\beta)^2}\right|. \tag{3.14}$$

By using triangle inequality, then

$$\left|a_3 - a_2^2\right| \le \frac{1}{12}\frac{\alpha\left(-\left(2(\alpha+\beta)^2 - 3\alpha(\alpha+2\beta)\right)c^2 + 2x\left(4(\alpha+\beta)^2 - c^2(\alpha+\beta)^2\right)\right)}{(\alpha+2\beta)(\alpha+\beta)^2} \tag{3.15}$$

and let $|x| = \phi$, thus

$$\begin{aligned}\left|a_3 - a_2^2\right| &\le \frac{1}{12}\frac{\alpha\left(-\left(2(\alpha+\beta)^2 - 3\alpha(\alpha+2\beta)\right)c^2 + 2\phi\left(4(\alpha+\beta)^2 - c^2(\alpha+\beta)^2\right)\right)}{(\alpha+2\beta)(\alpha+\beta)^2} \\ &= F(c,\phi).\end{aligned} \tag{3.16}$$

Further, differentiate $F(c,\phi)$ with respect to ϕ ,

$$\frac{\partial F}{\partial \phi} = \frac{1}{12}\frac{\alpha\left(8(\alpha+\beta)^2 - 2\left(\left(c^2\right)(\alpha+\beta)^2\right)\right)}{(\alpha+2\beta)(\alpha+\beta)^2}. \tag{3.17}$$

Observe that $\frac{\partial F}{\partial \phi} > 0$ for $c \in [0,2)$. Therefore, $F(c,\phi)$ increasing function of ϕ and $\max_{0\le\phi\le1} F(c,\phi) = F(c,1) = G(c)$. We get

$$G(c) = \frac{\left(3\alpha(\alpha+2\beta) - 4(\alpha+\beta)^2\right)c^2 + 8(\alpha+\beta)^2}{(\alpha+\beta)(\alpha+2\beta)^2}. \tag{3.18}$$

Then,

$$G'(c) = \frac{2\left(3\alpha(\alpha+2\beta) - 4(\alpha+\beta)^2\right)c}{(\alpha+2\beta)(\alpha+\beta)^2} \tag{3.19}$$

we observe that $G'(c) \le 0$ for every $c \in [0,2]$. Hence, $G(c)$ becomes an increasing function of c at maximum value $c = 0$. Substitute the value of c obtained in (3.18) and we get

$$G_{\max} = G(0) = \frac{8(\alpha+\beta)}{(\alpha+2\beta)^2}. \tag{3.20}$$

From (3.20), we obtain

$$\left|\frac{\alpha\left(2\left(c(\alpha+\beta)^2 + x\left(4(\alpha+\beta)^2 - c^2(\alpha+\beta)^2\right)\right) - 3\alpha c^2(\alpha+2\beta)\right)}{(\alpha+2\beta)(\alpha+\beta)^2}\right| \le \frac{8(\alpha+\beta)}{(\alpha+2\beta)^2}. \tag{3.21}$$

Therefore, simplifying (3.20) and (3.14) lead us to get

$$\left|a_3 - a_2^2\right| \le \frac{2}{3}\frac{(\alpha+\beta)}{(\alpha+2\beta)^2}.$$

This complete the proof of Theorem 3.2. □

To find $H_3(1)$, we substitute Lemma 2.5, Theorem 3.1, Theorem 3.2 and Lemma 2.4 into the inequality of (1.4) to obtain the following corollary.

Corollary 3.3. *For* $f \in B$, *then*

$$\left|H_3(1)\right| \le \frac{\alpha^2}{3}\left[\begin{array}{l}\dfrac{8\alpha}{9(\alpha+2\beta)^3} + \left(\dfrac{\alpha^2}{4(\alpha+3\beta)}\left(\dfrac{9T-4\alpha(\alpha+3\beta)}{3(3T-2(\alpha+3\beta))}\right)^{\frac{3}{2}}\right) \\ +\dfrac{4\alpha(\alpha+\beta)}{15(\alpha+4\beta)(\alpha+2\beta)^2}\end{array}\right]. \tag{3.22}$$

where $0<\alpha\le 1, 0\le\beta<1$ *and* $T=(\alpha+\beta)(\alpha+2\beta)$.

Theorem 3.4. *For* $f \in B$ *then*

$$\left|\Delta_1\right| \le \frac{\alpha^2\left(116\alpha^6+1624\alpha^5\beta+8331\alpha^4\beta^2+18628\alpha^3\beta^3+15127\alpha^2\beta^4-3326\alpha\beta^5-7368\beta^6\right)}{180T^2(\alpha+5\beta)(\alpha+4\beta)(\alpha+3\beta)^2},$$

$$\left|\Delta_2\right| \le \frac{\alpha^2\left(173\alpha^6+2768\alpha^5\beta+16481\alpha^4\beta^2+43406\alpha^3\beta^3+41168\alpha^2\beta^4-14752\alpha\beta^5-32160\beta^6\right)}{450T(\alpha+2\beta)(\alpha+4\beta)^2(\alpha+5\beta)(\alpha+3\beta)^2},$$

$$\left|\Delta_3\right| \le \frac{\alpha^2\begin{pmatrix}2340\alpha^8+56160\alpha^7\beta+546889\alpha^6\beta^2+2757986\alpha^5\beta^3+7500565\alpha^4\beta^4\\+9746052\alpha^3\beta^5+1575828\alpha^2\beta^6-8891424\alpha\beta^7-6257088\beta^8\end{pmatrix}}{5400T(\alpha+2\beta)(\alpha+4\beta)^2(\alpha+5\beta)(\alpha+3\beta)^3}$$

where $0<\alpha\le 1, 0\le\beta<1$ *and* $T=(\alpha+\beta)(\alpha+2\beta)$.

Proof. Let $f \in B$. From Lemma 2.3 where $p \in P$, this can easily get

$$n\left(\alpha+(n-1)\beta\right)a_n = c_{n-1} \tag{3.23}$$

By substituting (3.23) in (1.6) and note that $T=(\alpha+\beta)(\alpha+2\beta)$, it follows that

$$\Delta_1 = \frac{c_2c_5\alpha^2}{18(\alpha+2\beta)(\alpha+5\beta)} - \frac{c_3c_4\alpha^2}{20(\alpha+3\beta)(\alpha+4\beta)} - \frac{c_1^2c_5\alpha^3}{24(\alpha+\beta)^2(\alpha+5\beta)} + \frac{c_1c_2c_4\alpha^3}{30T(\alpha+4\beta)} + \frac{c_3^2c_1\alpha^3}{32(\alpha+3\beta)^2(\alpha+\beta)} - \frac{c_3c_2^2\alpha^3}{36(\alpha+2\beta)^2(\alpha+3\beta)}, \tag{3.24}$$

$$\Delta_2 = \frac{c_3c_5\alpha^2}{24(\alpha+3\beta)(\alpha+5\beta)} - \frac{c_4c_4\alpha^2}{25(\alpha+4\beta)^2} + \frac{c_1c_3c_4\alpha^3}{40(\alpha+\beta)(\alpha+3\beta)(\alpha+4\beta)} - \frac{c_1c_2c_5\alpha^3}{36T(\alpha+5\beta)} + \frac{c_2^2c_4\alpha^3}{45(\alpha+2\beta)^2(\alpha+4\beta)} - \frac{c_2c_3^2\alpha^3}{48(\alpha+2\beta)(\alpha+3\beta)^2} \tag{3.25}$$

$$\Delta_3 = \frac{c_1c_2c_5\alpha^3}{48T(\alpha+5\beta)} - \frac{c_1c_4^2\alpha^3}{50(\alpha+\beta)(\alpha+4\beta)^2} + \frac{c_2c_3c_4\alpha^3}{30(\alpha+2\beta)(\alpha+3\beta)(\alpha+4\beta)} - \frac{c_3^3\alpha^3}{64(\alpha+3\beta)^3} - \frac{c_2^2c_5\alpha^2}{54(\alpha+2\beta)^2(\alpha+5\beta)}. \tag{3.26}$$

Applying triangle inequality and using the sharp inequality for the function $\frac{1+z}{1-z}$ when $|c_n| \le 2$ for each $n = 1,2,3,...$ to get

$$|\Delta_1| \le -\frac{\alpha^2}{6(\alpha+5\beta)}\left(\frac{1}{(\alpha+2\beta)} - \frac{2\alpha}{(\alpha+\beta)^2}\right) - \frac{\alpha^2}{9(\alpha+3\beta)}\left(\frac{1}{(\alpha+4\beta)} - \frac{2\alpha}{(\alpha+2\beta)^2}\right) - \frac{\alpha^2}{8(\alpha+3\beta)}\left(\frac{1}{(\alpha+4\beta)} - \frac{2\alpha}{(\alpha+\beta)(\alpha+3\beta)}\right) - \frac{67\alpha^2}{360(\alpha+4\beta)}\left(\frac{1}{(\alpha+3\beta)} - \frac{2\alpha}{T}\right) - \frac{19\alpha^2}{360(\alpha+2\beta)}\left(\frac{1}{(\alpha+5\beta)} - \frac{2\alpha}{(\alpha+\beta)(\alpha+4\beta)}\right) + \frac{\alpha^2}{360(\alpha+2\beta)(\alpha+5\beta)},$$

$$|\Delta_2| \le -\frac{\alpha^2}{9(\alpha+5\beta)}\left(\frac{1}{(\alpha+3\beta)} - \frac{2\alpha}{T}\right) - \frac{4\alpha^2}{45(\alpha+4\beta)}\left(\frac{1}{(\alpha+4\beta)} - \frac{2\alpha}{(\alpha+2\beta)^2}\right) - \frac{\alpha^2}{12(\alpha+3\beta)}\left(\frac{1}{(\alpha+5\beta)} - \frac{2\alpha}{(\alpha+2\beta)(\alpha+3\beta)}\right) - \frac{16\alpha^2}{225(\alpha+4\beta)}\left(\frac{1}{(\alpha+4\beta)} - \frac{2\alpha}{(\alpha+\beta)(\alpha+3\beta)}\right) - \frac{26\alpha^2}{900(\alpha+3\beta)}\left(\frac{1}{(\alpha+5\beta)} - \frac{2\alpha}{(\alpha+\beta)(\alpha+4\beta)}\right) + \frac{\alpha^2}{900(\alpha+3\beta)(\alpha+5\beta)},$$

$$
\begin{aligned}
|\Delta_3| \le & -\frac{2\alpha^2}{27(\alpha+5\beta)}\left(\frac{1}{(\alpha+4\beta)}-\frac{2\alpha}{(\alpha+2\beta)^2}\right)-\frac{\alpha^2}{12(\alpha+5\beta)}\left(\frac{1}{(\alpha+4\beta)}-\frac{2\alpha}{(\alpha+\beta)(\alpha+3\beta)}\right) \\
& -\frac{\alpha^2}{16(\alpha+3\beta)}\left(\frac{1}{(\alpha+6\beta)}-\frac{2\alpha}{(\alpha+3\beta)^2}\right)-\frac{\alpha^2}{16(\alpha+3\beta)}\left(\frac{1}{(\alpha+6\beta)}-\frac{2\alpha}{(\alpha+4\beta)(\alpha+2\beta)}\right) \\
& -\frac{2\alpha^2}{25(\alpha+4\beta)}\left(\frac{1}{(\alpha+5\beta)}-\frac{2\alpha}{(\alpha+\beta)(\alpha+4\beta)}\right)-\frac{17\alpha^2}{240(\alpha+4\beta)}\left(\frac{1}{(\alpha+5\beta)}-\frac{2\alpha}{(\alpha+2\beta)(\alpha+3\beta)}\right) \\
& +\frac{\alpha^2}{10800(\alpha+4\beta)(\alpha+5\beta)}.
\end{aligned}
$$

With a simple simplification, we have

$$
|\Delta_1| \le \frac{\alpha^2\left(116\alpha^6+1624\alpha^5\beta+8331\alpha^4\beta^2+18628\alpha^3\beta^3+15127\alpha^2\beta^4-3326\alpha\beta^5-7368\beta^6\right)}{180(\alpha+2\beta)^2(\alpha+\beta)^2(\alpha+5\beta)(\alpha+4\beta)(\alpha+3\beta)^2},
$$

$$
|\Delta_2| \le \frac{\alpha^2\left(173\alpha^6+2768\alpha^5\beta+16481\alpha^4\beta^2+43406\alpha^3\beta^3+41168\alpha^2\beta^4-14752\alpha\beta^5-32160\beta^6\right)}{450(\alpha+2\beta)^2(\alpha+4\beta)^2(\alpha+5\beta)(\alpha+\beta)(\alpha+3\beta)^2},
$$

$$
|\Delta_3| \le \frac{\alpha^2\begin{pmatrix}2340\alpha^8+56160\alpha^7\beta+546889\alpha^6\beta^2+2757986\alpha^5\beta^3+7500565\alpha^4\beta^4 \\ +9746052\alpha^3\beta^5+1575828\alpha^2\beta^6-8891424\alpha\beta^7-6257088\beta^8\end{pmatrix}}{5400(\alpha+2\beta)^2(\alpha+4\beta)^2(\alpha+5\beta)(\alpha+\beta)(\alpha+3\beta)^3}.
$$

The proof is complete. □

Lastly, substituting the value of (3.22), $|\Delta_1|$, $|\Delta_2|$, $|\Delta_3|$ with the inequality from Lemma 2.4 and Lemma 2.5 into (1.5) and upon simplification, we get the following corollary.

Corollary 3.5. For $f \in B$, then

$$\begin{aligned}
&\left|H_4(1)\right| \\
&\le \frac{2\alpha^3}{21(\alpha+6\beta)}\left[\frac{8\alpha}{9(\alpha+2\beta)^3}+\left(\frac{\alpha^2}{4(\alpha+3\beta)}\left(\frac{9T-4\alpha(\alpha+3\beta)}{3(3T-2(\alpha+3\beta))}\right)^{\frac{3}{2}}\right)+\frac{4\alpha(\alpha+\beta)}{15(\alpha+4\beta)(\alpha+2\beta)^2}\right] \\
&\quad+\frac{\alpha^3\left(116\alpha^6+1624\alpha^5\beta+8331\alpha^4\beta^2+18628\alpha^3\beta^3+15127\alpha^2\beta^4-3326\alpha\beta^5-7368\beta^6\right)}{540(\alpha+5\beta)^2(\alpha+2\beta)^2(\alpha+\beta)^2(\alpha+4\beta)(\alpha+3\beta^2)} \qquad (3.27)\\
&\quad+\frac{\alpha^3\left(173\alpha^6+2768\alpha^5\beta+16481\alpha^4\beta^2+43406\alpha^3\beta^3+41168\alpha^2\beta^4-14752\alpha\beta^5-32160\beta^6\right)}{11253(\alpha+2\beta)^2(\alpha+4\beta)^3(\alpha+5\beta)(\alpha+\beta)(\alpha+3\beta)^2} \\
&\quad+\frac{\alpha^3\begin{pmatrix}2340\alpha^8+56160\alpha^7\beta+546889\alpha^6\beta^2+2757986\alpha^5\beta^3+7500565\alpha^4\beta^4\\+9746052\alpha^3\beta^5+1575828\alpha^2\beta^6-8891424\alpha\beta^7-6257088\beta^8\end{pmatrix}}{10800(\alpha+2\beta)^2(\alpha+4\beta)^2(\alpha+5\beta)(\alpha+\beta)(\alpha+3\beta)^4}.
\end{aligned}$$

where $0<\alpha\le 1, 0\le\beta<1$ *and* $T=(\alpha+\beta)(\alpha+2\beta)$.

Remark 3.6. From Arif *et al.* (2018), $\left|H_4(1)\right| \le \frac{73757}{94500} \simeq 0.7805$ coincides with (3.27) by choosing $\alpha = 1$ and $\beta = 0$.

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