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3 PROOFS OF 2 WELL-KNOWN THEOREMS ON STARLIKE AND CONVEX FUNCTIONS

(3 Bukti bagi 2 Teorem yang Dikenali bagi Fungsi Bakbintang dan Cembung)

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ABSTRACT

Let f be analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. We give three different proofs for the well-known sharp bounds for the second Hankel determinant $|H_2(2)(f)| = |a_2a_4 - a_3^2|$ for starlike and convex functions.

Keywords: analytic; univalent; starlike; convex; Hankel determinant; coefficients

ABSTRAK

Andaikan f analisis dalam $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, dan diberi oleh $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Tiga pembuktian berbeza bagi batas terbaik bagi penentu kedua Hankel $|H_2(2)(f)| = |a_2a_4 - a_3^2|$ diberi untuk fungsi bakbintang dan cembung.

Kata kunci: analisis; univalen; bakbintang; cembung; penentu Hankel; pekali

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions f analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with Taylor series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$, consisting of univalent (i.e. one-to-one) functions. The famous Bieberbach conjecture of 1916 asserted that if $f \in \mathcal{S}$, then $|a_n| \leqslant n$ for $n \geqslant 2$, with equality when $f(z) = z/(1-z)^2$.

This was eventually proved by de Branges in 1985.

A function $f \in \mathcal{A}$ is called starlike (with respect to the origin) if $f(\mathbb{D})$ is starlike with respect to the origin. We denote this class of functions by $\mathcal{S}^*$, and note it is well-known that a function $f \in \mathcal{A}$ belongs to $\mathcal{S}^*$ if, and only if, for $z \in \mathbb{D}$,

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0. \tag{2}$$

A function $f \in \mathcal{A}$ is called convex if $f(\mathbb{D})$ is convex. We denote this class of functions by $\mathcal{C}$, and note that it is well-known that a function $f \in \mathcal{A}$ belongs to $\mathcal{C}$ if, and only if, for $z \in \mathbb{D}$,

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0. \tag{3}$$

A great deal of effort has been made in recent years by many authors to find the sharp bound for the second Hankel determinant $|H_2(2)(f)| = |a_2a_4 - a_3^2|$, for a variety of subclasses of $\mathcal{S}$.

The following results were proved in (Janteng et.al 2007).

Let $f \in \mathcal{S}^*$ and be given by Eq. (1), then $|H_2(2)(f)| \leqslant 1$, and if $f \in \mathcal{C}$, then

$$|H_2(2)(f)| \leqslant \frac{1}{8}.$$

Both inequalities are sharp.

In this paper we give three different proofs of these two results which demonstrates 3 important methods of attacking problems when considering functionals in subclasses of functions of $\mathcal{S}$.

We first define the class $\mathcal{P}$ of functions with positive real part in $\mathbb{D}$.

Definition 1.1. *A function $p \in \mathcal{P}$ if, and only if, for $z \in \mathbb{D}$, $\operatorname{Re} p(z) > 0$ and p is given by*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{4}$$

At this point we note that both the classes $\mathcal{S}^*$ and $\mathcal{C}$, and the functional $H_2(2)(f)$ are rotationally invariant. This means for example that if $f \in \mathcal{S}^*$, then $f_\theta(z) := e^{-i\theta} f(e^{i\theta}z) \in \mathcal{S}^*$ for $\theta \in \mathbb{R}$, and that

$$|a_2a_4 - a_3^2| = |a_2e^{i\theta}a_4e^{3i\theta} - (a_3e^{2i\theta})^2| = |a_2a_4 - a_3^2|.$$

We will use this idea in the following proofs to normalise the coefficient c_1 so that $c_1 := c \geqslant 0$, but since $|c_1| \leqslant 2$ (from Definition 1.1), we can assume that $0 \leqslant c \leqslant 2$.

2. $H_2(2)(f)$ for Starlike Functions

Next first note from Eq. (2) and Eq. (4), that we can write $zf'(z) = f(z)p(z)$, where $p \in \mathcal{P}$, and so equating coefficients we obtain, after a little computation

$$H_2(2)(f) = \left|\frac{c_1c_3}{3} - \frac{c_2^2}{4} - \frac{c_1^4}{12}\right|. \tag{5}$$

Thus we need to show Eq. (5) is bounded by 1.

We first give the original proof that $H_2(2)(f) \leqslant 1$ when $f \in \mathcal{S}^*$ due to Janteng et.al (2007), which uses the following lemma of Libera and Zlotkiewicz (1983).

Lemma 2.1. *Let $p \in \mathcal{P}$ and be of the form Eq. (4) with $c_1 \geqslant 0$, then*

$$2c_2 = c_1^2 + \zeta(4 - c_1^2)$$

and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)\zeta - c_1(4 - c_1^2)\zeta^2 + 2(4 - c_1^2)(1 - |\zeta|^2)\eta$$

for some ζ, $\eta \in \mathbb{C}$ such that $|\zeta| \leqslant 1$ and $|\eta| \leqslant 1$.

Proof 1

This is due to Janteng et. al (2007).

Proof. We use the above lemma to express the coefficients c_2 and c_3 in terms of c_1 to obtain

$$\begin{aligned}\frac{c_1c_3}{3}-\frac{c_2^2}{4}-\frac{c_1^4}{12}&=-\frac{c_1^4}{16}+\frac{c_1^2}{24}(4-c_1^2)\zeta-\frac{c_1^2}{12}(4-c_1^2)\zeta^2-\frac{1}{16}(4-c_1^2)^2\zeta^2\\&+\frac{c_1}{6}(4-c_1^2)(1-|\zeta|^2)\eta.\end{aligned}$$

We now use the rotationally invariant property to write $c_1 = c$ where $0 \leqslant c \leqslant 2$, and taking the modulus with $|\eta| \leqslant 1$, we obtain:

$$\begin{aligned}|H_2(2)(f)|\leqslant\frac{c^4}{16}&+\frac{c^2}{24}(4-c^2)|\zeta|+\frac{c^2}{12}(4-c^2)|\zeta|^2+\frac{1}{16}(4-c^2)^2|\zeta|^2\\&+\frac{c}{6}(4-c^2)(1-|\zeta|^2).\end{aligned}\tag{6}$$

Thus we must find the maximum value of this expression over $[0, 2] \times [0, 1]$.

Suppose that there exist a critical point $(c_0, |\zeta_0|)$ inside $[0, 2] \times [0, 1]$, then differentiating the right hand side of Eq. (6) with respect to $|\zeta|$ and equating to zero gives $c_0 = 2$, which is a contradiction. Thus any critical points must be on the edges of $[0, 2] \times [0, 1]$.

When $c = 0$, we obtain $|H_2(2)(f)| \leqslant |\zeta|^2 \leqslant 1$.

When $c = 2$, we obtain $|H_2(2)(f)| = 1$.

When $|\zeta| = 0$, we obtain $|H_2(2)(f)| \leqslant \dfrac{c^4}{16} + \dfrac{c}{6}(4 - c^2) \leqslant 1$, when $0 \leqslant c \leqslant 2$.

Finally when $|\zeta| = 1$, we obtain $|H_2(2)(f)| = 1$ once more.

Thus we have shown that $|H_2(2)(f)| \leqslant 1$ as required. □

Proof 2

Here we use a recent lemma of Cho, Kowalczyk and Lecko (2019) as follows.

Lemma 2.2. *Let $p \in \mathcal{P}$, and be of the form Eq. (4). Then*

$$c_1 = 2\zeta_1,$$

$$c_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2,$$

and

$$c_3 = 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)\bar{\zeta}_1\zeta_2^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3,$$

for some $\zeta_i \in \bar{\mathbb{D}}$, $i = 1, 2, 3$.

Proof. As before we need to consider

$$\frac{c_1c_3}{3}-\frac{c_2^2}{4}-\frac{c_1^4}{12},$$

but this time we use Lemma 2.2 to obtain

$$\begin{aligned}\frac{c_1c_3}{3}-\frac{c_2^2}{4}-\frac{c_1^4}{12}=&\frac{4\zeta_1^4}{3}+\frac{1}{4}(2\zeta_1^2+2(1-\zeta_1^2)\zeta_2)^2\\&-\frac{2}{3}\zeta_1(2\zeta_1^3+4(1-\zeta_1^2)\bar{\zeta}_1\zeta_2-2\zeta_1(1-\zeta_1^2)\zeta_2^2\\&+2(1-|\zeta_1|^2)(1-|\zeta_2|^2))\zeta_3.\end{aligned}\tag{7}$$

We now use the rotational property to normalise $c_1=2\zeta_1$, and write $\zeta_1=\zeta$, where now $0\leqslant\zeta\leqslant 1$.

Taking the modulus in Eq. (7), using the fact that $|\zeta_3|\leqslant 1$ and simplifying, we obtain

$$|H_2(2)(f)|\leqslant\zeta^4+\frac{2}{3}\zeta^2(1-\zeta^2)|\zeta_2|+\frac{1}{3}(3+\zeta^2)(1-\zeta^2)|\zeta_2|^2+\frac{4}{3}\zeta(1-\zeta^2)(1-|\zeta_2|^2).$$

Thus we need to find the maximum of the right hand side of the above expression, this time over $[0,1]\times[0,1]$.

As before, we use a contradiction argument to show that there are no critical points inside $[0,1]\times[0,1]$, this time differentiating with respect to $|\zeta_2|$.

Thus we are left to check the values on each of the edges of $[0,1]\times[0,1]$.

When $\zeta=0$, we have $|H_2(2)(f)|\leqslant|\zeta_2|^2\leqslant 1$.

When $\zeta=1$, we have $|H_2(2)(f)|=1$.

When $|\zeta_2|=0$, we have $|H_2(2)(f)|\leqslant\zeta^4+\frac{4}{3}\zeta(1-\zeta^2)\leqslant 1$, when $0\leqslant\zeta\leqslant 1$.

Finally when $|\zeta_2|=1$, we again have $|H_2(2)(f)|=1$. □

We note that the lemmas in the above two proofs are similar, which lead to solutions of a similar nature.

Proof 3

We use the following lemma of Choi, Kim and Sugawa (2007), which can often be useful to find sharp bounds in more difficult problems when the above two methods fail.

Lemma 2.3. *For $\zeta\in\bar{\mathbb{D}}$ suppose that A, B, C, are real numbers, and let*

$$Y(A,B,C)=\max\left\{|A+B\zeta+C\zeta^2|+1-|\zeta|^2:\zeta\in\bar{\mathbb{D}}\right\}.$$

If $AC\geqslant 0$, then

$$Y(A,B,C)=\begin{cases}|A|+|B|+|C|, & |B|\geqslant 2(1-|C|),\\ 1+|A|+\dfrac{B^2}{4(1-|C|)}, & |B|<2(1-|C|).\end{cases}$$

If $AC < 0$, *then*

$$Y(A,B,C) = \begin{cases} 1-|A|+\dfrac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1) \leqslant B^2 \wedge |B| < 2(1-|C|), \\ 1+|A|+\dfrac{B^2}{4(1+|C|)}, & B^2 < \min\left\{4(1+|C|)^2, -4AC(C^{-2}-1)\right\}, \\ R(A,B,C), & \text{otherwise}, \end{cases}$$

where

$$R(A,B,C) = \begin{cases} |A|+|B|-|C|, & |C|(|B|+4|A|) \leqslant |AB|, \\ -|A|+|B|+|C|, & |AB| \leqslant |C|(|B|-4|A|), \\ (|C|+|A|)\sqrt{1-\dfrac{B^2}{4AC}}, & \text{otherwise}. \end{cases}$$

Proof. We first use Lemma 2.1 to obtain after rotating c_1 so that $c_1 = c$ and $0 \leqslant c \leqslant 2$,

$$\begin{aligned} \frac{c_1c_3}{3} - \frac{c_2^2}{4} - \frac{c_1^4}{12} = &-\frac{c^4}{16} + \frac{c^2}{24}(4-c^2)\zeta - \frac{c^2}{12}(4-c^2)\zeta^2 - \frac{1}{16}(4-c^2)^2\zeta^2 \\ &+ \frac{c}{6}(4-c^2)(1-|\zeta|^2)\eta. \end{aligned} \tag{8}$$

Note first that when $c = 0$ in (2.4), we obtain $-\zeta^2$, so that $|H_2(2)(f)| \leqslant 1$. Also when $c = 2$, (2.4) becomes -1, and so again $|H_2(2)(f)| \leqslant 1$.

Thus we can assume that $c \neq 0$ and $c \neq 2$.

We now rewrite the right hand side of Eq. (8) as

$$\frac{c}{6}(4-c^2)(A+B\zeta+C\zeta^2+(1-|\zeta|^2)\eta), \tag{9}$$

where

$$A = -\frac{3c^3}{8(4-c^2)}, \quad B = \frac{c}{4}, \quad \text{and} \quad C = -\frac{12+c^2}{8c}.$$

We now apply Lemma 2.3.

Note that when $0 < c < 2$, $AC \geqslant 0$, and $|B| \geqslant 2(1-|C|)$, thus by Lemma 2.3, since $|\eta| \leqslant 1$, we have shown that

$$\max\left\{|A+B\zeta+C\zeta^2|+1-|\zeta|^2\right\} = \frac{6}{c(4-c^2)}.$$

Thus since $0 < c < 2$, Eq. (9) gives

$$|H_2(2)(f)| \leqslant \frac{c}{6}(4-c^2)\frac{6}{c(4-c^2)} = 1. \quad \square$$

3. $H_2(2)(f)$ for Convex Functions

From Eq. (3), we can write

$$1 + \frac{zf''(z)}{f'(z)} = p(z)$$

for some $p \in \mathcal{P}$, and equating coefficients obtain

$$H_2(2)(f) = \frac{1}{144}\Big(-c_1^4 + c_1^2c_2 - 4c_2^2 + 6c_1c_3\Big). \tag{10}$$

Proof 1

We give the original proof given by Janteng et.al (2007).

Proof. From Eq. (10) and Lemma 2.1 after rotation we obtain with $0 \leqslant c \leqslant 2$, and writing $|\zeta| = \rho$,

$$\begin{aligned}\Big| - c_1^4 + c_1^2 c_2 - 4c_2^2 + 6c_1c_3\Big| &= (4-c^2)\Big|\frac{3c^2\zeta}{2} - \frac{(8+c^2)\zeta^2}{2} + 3c(1-|\zeta|^2)\eta\Big| \\ &\leqslant (4-c^2)\Big(3c\rho + \frac{(c-4)(c-2)\rho^2}{2}\Big) \\ &= F(c,\rho) \text{ say,}\end{aligned} \tag{11}$$

and a simple exercise in elementary calculus shows that $F(c, \rho)$ has maximum value 18 on $0 < c < 2$ when $c = 1$ and $\rho = 1$, and noting that when $c = 2$ the value is 0, and at $c = 0$ is 16. This completes the proof. □

Proof 2

The proof is a simpler version of Proof 1.

Proof. Consider again Eq. (11), and use the triangle inequality to obtain, writing $|\zeta| = \rho$

$$\Big| - c_1^4 + c_1^2 c_2 - 4c_2^2 + 6c_1c_3\Big| \leqslant (4-c^2)\Big(\frac{3c^2\rho}{2} + \frac{(8+c^2)\rho^2}{2} + 3c(1-\rho^2)\Big) = G(c,\rho) \text{ say.}$$

A simple exercise shows that there are no critical points of $G(c, \rho)$ inside the rectangle $[0, 2] \times [0, 1]$, and on the edges we have $G(0, \rho) = 16\rho^2 \leqslant 16,\ G(2, \rho) = 0,\ G(c, 0) = 3c(4-c^2) < 18$, and $G(c, 1) = 18$, which completes the proof. □

Proof 3

Proof. Consider again Eq. (11), first noting again that when $c = 0$ we obtain 16, and when $c = 2$ we obtain 0. Now write

$$\frac{1}{144}\Big| - c_1^4 + c_1^2 c_2 - 4c_2^2 + 6c_1c_3\Big| = 3c(4-c^2)\Big|\frac{c\zeta}{2} - \frac{(8+c^2)\zeta^2}{6c} + (1-|\zeta|^2)\eta\Big|,$$

and use Lemma 2.3, with $A = 0$, $B = \dfrac{c}{2}$ and $C = -\dfrac{8+c^2}{6c}$ and so $AC \geqslant 0$.

A simple exercise shows that the condition $|B| \geqslant 2(1 - |C|)$ is satisfied when $0 < c < 2$, which gives the bound $|A| + |B| + |C| = \dfrac{2(2+c^2)}{3c}$ and so

$$|H_2(2)(f)| \leqslant \frac{1}{144} 3c(4-c^2)\frac{2(2+c^2)}{3c} = \frac{1}{72}(4-c^2)(2+c^2),$$

which has max $\dfrac{1}{8}$. This completes the proof. □

4. Extreme Functions

We end by noting that the inequality $|H_2(2)(f)| \leqslant 1$ for starlike functions is sharp for the Koebe function, and that inequality $|H_2(2)(f)| \leqslant \dfrac{1}{8}$ for convex functions is sharp when

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1-z^2}{1-z+z^2}.$$

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