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APPROXIMATION THEORY IN *UP*-ALGEBRAS BASED ON INTUITIONISTIC FUZZY SETS

(Teori Penghampiran dalam Aljabar-*UP* Berdasarkan Set Kabur Berintuisi)

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ABSTRACT

This paper discusses the upper and lower approximations of Atanassov intuitionistic fuzzy sets in crisp and fuzzy approximation spaces in which both constructive and axiomatic approaches are used. In the constructive approach, concepts of intuitionistic fuzzy sets are defined, properties of intuitionistic fuzzy approximation operators are examined. Different classes of intuitionistic fuzzy set algebras are obtained from different types of fuzzy relations. In the axiomatic approach, an operator-oriented characterization of intuitionistic fuzzy sets is proposed.

Keywords: *UP*-algebra; intuitionistic fuzzy set

ABSTRAK

Kertas kerja ini membincangkan penghampiran atas dan bawah bagi set kabur intuisi Atanassov dalam ruang penghampiran yang jelas dan kabur di mana kedua-dua pendekatan konstruktif dan aksiomatik digunakan. Dalam pendekatan konstruktif, konsep set kabur intuisi ditakrifkan, sifat pengendali penghampiran kabur intuisi diperiksa. Kelas algebra set kabur intuisi yang berbeza diperoleh daripada jenis hubungan kabur yang berbeza. Dalam pendekatan aksiomatik, pencirian berorientasikan operator bagi set kabur intuisi dicadangkan.

Kata kunci: *UP*-aljabar; set kabur intuisi

1. Introduction

People are facing critical problems involving uncertainty and impreciseness in everyday life. Inexact and incomplete information data has not complete and precise reasoning possibly. Nowadays, the gap between the world full of vagueness and traditional mathematics with precise concepts is going to reduce with highly appreciated mode. Researchers are keen to deal with different nature of uncertainty with different methods. In this regard, they studied many newly defined theories such as Rough Sets (RSs), Fuzzy Sets (FSs), Intuitionistic Fuzzy Sets (IFSs), and Soft Sets (SSs).

The FSs are very important to manage the uncertainties in real-world problems. They have numerous applications in several fields such as vacuum cleaner, control of subway systems, unmanned helicopters, transmission systems, models for new pricing, and weather forecasting systems. This logic has given new life to scientific fields and has been used in many fields such as electronics, image processing, and optimization.

The IFS theory introduced by Atanassov (1986) is one of the generalizations of FS theory. In IFS, membership degree, degree of nonmembership, and degree of hesitation of every element are expressed whose sum must be equal to 1. As we know, IFSs have degree of membership and degree of nonmembership which are more valuable in medical field. Intuitionistic fuzzy environment is more suitable to diagnose disease than FS due to its nonmembership degree. The FSs have only degree of membership but IF soft sets (IFSSs) keep a controlled degree of the vagueness, and they transform an imprecise pattern classification problem into a well-defined and precise optimization problem because an IFS gives out the uncertainty by a nonmembership degree.

In real-life problems, IF logic has more effective use to control and overcome the uncertainty than fuzzy logic. Iampan introduced the concept of *UP*-algebras (see Iampan (2017)) as a generalization of *KU*-algebras (see Mostafa et al. (2011)) and investigated their properties. Later, several substructures of *UP*-algebras have been discussed by several researchers (see Iampan (2019); Jun et al. (2021); Satirad et al. (2021)). Somjanta et al. applied the concept of fuzzy sets to *UP*-algebras and investigated various properties (see Somjanta et al. (2016)). Kesorn et al. Kesorn et al. (2015) applied the concept of intuitionistic fuzzy sets to *UP*-algebras and investigated various properties. The aim of this paper is to study upper and lower approximations of intuitionistic fuzzy sets in *UP*-algebra. Before begin, the definition of *UP*-algebras is reviewed.

2. Preliminaries

An algebra $X = (X, *, 0)$ of type $(2, 0)$ is called a *UP-algebra* (see Iampan (2017)) if it satisfies the following conditions:

$$(i)\ \{\forall x, y, z \in X\}, ((y * z) * ((x * y) * (x * z)) = 0), \tag{1}$$
$$(ii)\ \{\forall x \in X\}, (0 * x = x), \tag{2}$$
$$(iii)\ \{\forall x \in X\}, (x * 0 = 0), \tag{3}$$
$$(iv)\ \{\forall x, y \in X\}, (x * y = 0 = y * x \ \Rightarrow\ x = y). \tag{4}$$

If X satisfies Eqs. (2), (3), (4), and

$$\{\forall x, y, z \in X\}, ((x * y) * ((y * z) * (x * z)) = 0), \tag{5}$$

then, X is a *KU-algebra*. A binary relation " $\leq$ " on a *UP*-algebra X is defined as follows:

$$\{\forall x, y \in X\}, (\ x \leq y \ \Leftrightarrow\ x * y = 0\ ). \tag{6}$$

In a *UP*-algebra X, the following assertions are valid (see Iampan (2017)).

$$(i)\ \{\forall x \in X\}, (x * x = 0), \tag{7}$$
$$(ii)\ \{\forall x, y, z \in X\}, (x * y = 0,\ y * z = 0 \ \Rightarrow\ x * z = 0), \tag{8}$$
$$(iii)\ \{\forall x, y, z \in X\}, (x * y = 0 \ \Rightarrow\ (z * x) * (z * y) = 0), \tag{9}$$
$$(iv)\ \{\forall x, y, z \in X\}, (x * y = 0 \ \Rightarrow\ (y * z) * (x * z) = 0), \tag{10}$$
$$(v)\ \{\forall x, y \in X\}, (x * (y * x) = 0), \tag{11}$$
$$(vi)\ \{\forall x, y \in X\}, ((y * x) * x = 0 \ \Leftrightarrow\ x = y * x), \tag{12}$$
$$(vii)\ \{\forall x, y \in X\}, (x * (y * y) = 0). \tag{13}$$

Every *KU*-algebra X satisfies:

$$\{\forall x, y, z \in X\}, (x * (y * z) = y * (x * z)). \tag{14}$$

A nonempty subset A of a *UP*-algebra X is called a *UP-subalgebra* of X (see Iampan (2017)) if it satisfies the following condition:

$$\{\forall x, y \in X\}, (x \in A,\ y \in A \ \Rightarrow\ x * y \in A). \tag{15}$$

A subset F of a *UP*-algebra X is called

- A *UP-filter* of X (see Somjanta et al. (2016)) if it satisfies the following conditions:

$$(i)\, 0 \in F, \tag{16}$$
$$(ii)\, \{\forall x, y \in X\}, (x \in F,\ x * y \in F \ \Rightarrow\ y \in F). \tag{17}$$

- A *near UP-filter* of X (see Somjanta et al. (2016)) if it satisfies:

$$\{\forall x, y \in X\}, (y \in F \ \Rightarrow\ x * y \in F). \tag{18}$$

Definition 2.1 (Atanassov (1986))**.** Let X be a nonempty set. An *intuitionistic fuzzy set* on X is defined to be a structure

$$\mathcal{C}_X := \{\langle x, f(x), g(x)\rangle \mid x \in X\}, \tag{19}$$

where $f : X \to [0, 1]$ is the degree of membership of x to $\mathcal{C}$ and $g : X \to [0, 1]$ is the degree of non-membership of x to $\mathcal{C}$ such that $0 \leq f(x) + g(x) \leq 1$.

Hereafter, we use $f(x)$ and $g(x)$ instead of $(f(x))$ and $(g(x))$, respectively, and the intuitionistic fuzzy set in Eq. (19) is simply denoted by $\mathcal{C}_X := (X, f, g)$. Next, we state the following definition given by Guntasow et al. (2017); Iampan (2017, 2019); Somjanta et al. (2016).

Definition 2.2. A intuitionistic fuzzy set $\mathcal{C}_X := (X, f, g)$ on X is called

- An *intuitionistic fuzzy UP-subalgebra* of X if it satisfies:

$$\{\forall x, y \in X\}, \begin{pmatrix} f(x * y) \geq \min\{f(x), f(y)\} \\ g(x * y) \leq \max\{g(x), g(y)\} \end{pmatrix}. \tag{20}$$

- An *intuitionistic fuzzy near UP-filter* of X if it satisfies:

$$\{\forall x, y \in X\}, \begin{pmatrix} f(x * y) \geq f(y) \\ g(x * y) \leq g(y) \end{pmatrix}. \tag{21}$$

- An *intuitionistic fuzzy UP-filter* of X if it satisfies:

$$\{\forall x \in X\}, (f(0) \geq f(x),\ g(0) \leq g(x)), \tag{22}$$
$$\{\forall x, y \in X\}, \begin{pmatrix} f(y) \geq \min\{f(x * y), f(x)\} \\ g(y) \leq \max\{g(x * y), g(x)\} \end{pmatrix}. \tag{23}$$

- An *intuitionistic fuzzy UP-ideal* of X if it satisfies:

$$\{\forall x, y, z \in X\}, \begin{pmatrix} f(x * z) \geq \min\{f(x * (y * z)), f(y)\} \\ g(x * z) \leq \max\{g(x * (y * z)), g(y)\} \end{pmatrix}. \tag{24}$$

- An *intuitionistic fuzzy strong UP-ideal* of X if it satisfies:

$$\{\forall x, y \in X\}, \begin{pmatrix} f(x) \geq \min\{f((z * y) * (z * x)), f(y)\} \\ g(x * z) \leq \max\{g((z * y) * (z * x)), g(y)\} \end{pmatrix}. \tag{25}$$

Theorem 2.3 (Thongngam and Iampan (2019))**.** *An intuitionistic fuzzy set $\mathcal{C}_X := (X, f, g)$ is a intuitionistic fuzzy strong UP-ideal if and only if it is constant.*

3. Approximations of Intuitionistic Fuzzy Sets in UP-Algebras

Let X be a set and ρ an equivalence relation on X. If $x \in X$, then the ρ-class of X is the set $(x)_\rho$ defined as follows: $(x)_\rho = \{y \in X : (x,y) \in \rho\}$. An equivalence relation ρ on a *UP*-algebra $X = (X,*,0)$ is said to be a congruence relation if $\{\forall x,y,z \in X\}, ((x,y) \in \rho)(x*z, y*z) \in \rho, (z*x, z*y) \in \rho)$.

Definition 3.1. For nonempty subsets A and B of a *UP*-algebra $X = (X,*,0)$, denote $AB = A*B = \{a*b : a \in A \text{ and } b \in B\}$. If ρ is a congruence on a *UP*-algebra $X = (X,*,0)$, then $\{\forall x,y \in X\}, ((x)_\rho(y)_\rho(x*y)_\rho)$. A congruence relation ρ on a *UP*-algebra $X = (X,*,0)$ is said to be complete if $\{\forall x,y \in X\}, ((x)_\rho(y)_\rho = (x*y)_\rho)$.

Definition 3.2. Let ρ be an equivalence relation on a nonempty set X and $C = (f,g)$ an intuitionistic fuzzy set in X. Then

(1) the upper approximation of C is defined by

$$\rho^+(C) = \{x, \overline{f}(x), \overline{g}(x)) : x \in X\},$$

where $\overline{f}(x) = \sup\limits_{a \in (x)_\rho}\{f(a)\}$ and $\overline{g}(x) = \inf\limits_{a \in (x)_\rho}\{g(a)\}$.

(2) the lower approximation of C is defined by

$$\rho^-(C) = \{(x, \underline{f}(x), \underline{g}(x)) : x \in X\},$$

where $\underline{f}(x) = \inf\limits_{a \in (x)_\rho}\{f(a)\}$ and $\underline{g}(x) = \sup\limits_{a \in (x)_\rho}\{g(a)\}$.

Theorem 3.3. *Let ρ be an equivalence relation on a nonempty set X and $C = (f,g)$ an intuitionistic fuzzy set in X. Then the following statements hold:*

(1) $\rho^+(C)$ is an intuitionistic fuzzy set in X,
(2) $\rho^-(C)$ is an intuitionistic fuzzy set in X.

Proof. Let $x \in X$.
(1). Consider

$$\begin{aligned} 0 &\leq \overline{f}(x) + \overline{g}(x) \\ &= \sup_{a \in (x)_\rho}\{f(a)\} + \inf_{a \in (x)_\rho}\{g(a)\} \\ &\leq \sup_{a \in (x)_\rho}\{f(a)\} + \inf_{a \in (x)_\rho}\{1 - f(a)\} \\ &= \sup_{a \in (x)_\rho}\{f(a)\} + 1 - \sup_{a \in (x)_\rho}\{f(a)\} \\ &= 1. \end{aligned}$$

Hence, $0 \leq \overline{f}(x) + \overline{g}(x) \leq 1$. Therefore, $\rho^+(C)$ is an intuitionistic fuzzy set in X.
(2). Similar to the proof of (1). □

Lemma 3.1. *If ρ is an equivalence relation on a nonempty set X and $C = (f,g)$ an intuitionistic fuzzy set in X, then*

$$\{\forall x,y \in X\}, \left(x\rho y \Rightarrow \begin{cases} \overline{f}(x) = \overline{f}(y), \overline{g}(x) = \overline{g}(y), \\ \underline{f}(x) = \underline{f}(y), \underline{g}(x) = \underline{g}(y). \end{cases} \right). \tag{26}$$

Proof. Let $x, y \in X$ be such that $x\rho y$. Then

$$\overline{f}(x) = \sup_{a\in(x)_\rho}\{f(a)\} = \sup_{b\in(y)_\rho}\{f(b)\} = \overline{f}(y),$$

$$\overline{g}(x) = \inf_{a\in(x)_\rho}\{g(a)\} = \inf_{b\in(y)_\rho}\{g(b)\} = \overline{g}(y),$$

$$\underline{f}(x) = \inf_{a\in(x)_\rho}\{f(a)\} = \inf_{b\in(y)_\rho}\{f(b)\} = \underline{f}(y),$$

$$\underline{g}(x) = \sup_{a\in(x)_\rho}\{g(a)\} = \sup_{b\in(y)_\rho}\{g(b)\} = \underline{g}(y). \quad \square$$

Theorem 3.4. *Let ρ be an congruence relation on a UP-algebra $X = (X, *, 0)$ and $C = (f, g)$ an intuitionistic fuzzy set in X. Then the following statements hold:*

(1) If C is an intuitionistic fuzzy UP-subalgebra of X and ρ is complete, then $\rho^-(C)$ is an intuitionistic fuzzy UP-subalgebra of X,
(2) If C is an intuitionistic fuzzy near UP-filter of X and ρ is complete, then $\rho^-(C)$ is an intuitionistic fuzzy near UP-filter of X,
(3) If C is an intuitionistic fuzzy UP-filter of X and $(0)_\rho = \{0\}$, then $\rho^-(C)$ is an intuitionistic fuzzy UP-filter of X,
(4) If C is an intuitionistic fuzzy UP-ideal of X, $(0)_\rho = \{0\}$, and ρ is complete, then $\rho^-(C)$ is an intuitionistic fuzzy UP-ideal of X,
(5) If C is an intuitionistic fuzzy strong UP-ideal of X, then $\rho^-(C)$ is an intuitionistic fuzzy strong UP-ideal of X.

Proof. (1). Assume that C is an intuitionistic fuzzy *UP*-subalgebra of X and ρ is complete. Then for all $x, y \in X$,

$$\begin{aligned}
\underline{f}(x*y) &= \inf_{c\in(x*y)_\rho}\{f(c)\} \\
&= \inf_{c\in(x)_\rho(y)_\rho}\{f(c)\} \\
&= \inf_{a*b\in(x)_\rho(y)_\rho}\{f(a*b)\} \\
&\geq \inf_{a\in(x)_\rho, b\in(y)_\rho}\{\min\{f(a), f(b)\}\} \text{ by } Eq. \text{ (20)} \\
&= \min\{\inf_{a\in(x)_\rho}\{f(a)\}, \inf_{b\in(y)_\rho}\{f(b)\} \\
&= \min\{\underline{f}(x), \underline{f}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\underline{g}(x*y) &= \sup_{c\in(x*y)_\rho}\{g(c)\} \\
&= \sup_{c\in(x)_\rho(y))_\rho}\{g(c)\} \\
&= \sup_{a*b\in(x)_\rho(y)_\rho}\{g(a*b)\} \\
&\leq \sup_{a\in(x)_\rho, b\in(y)_\rho}\{\max\{g(a), g(b)\}\} \text{ by } Eq. \text{ (20)} \\
&= \max\{\sup_{a\in(x)_\rho}\{g(a)\}, \sup_{b\in(y)_\rho}\{g(b)\} \\
&= \max\{\underline{g}(x), \underline{g}(y)\}.
\end{aligned}$$

Hence, $\rho^-(C)$ is an intuitionistic fuzzy *UP*-subalgebra of X.

(2). Assume that C is an intuitionistic fuzzy near *UP*-filter of X and ρ is complete. Then for all $x, y \in X$,

$$\begin{aligned}
\underline{f}(x*y) &= \inf_{c\in(x*y)_\rho}\{f(c)\}\\
&= \inf_{c\in(x)_\rho(y)_\rho}\{f(c)\}\\
&= \inf_{a*b\in(x)_\rho(y)_\rho}\{f(a*b)\}\\
&\geq \inf_{b\in(y)_\rho}\{f(b)\}\\
&= \underline{f}(y) \ by\ Eq.\ (21)
\end{aligned}$$

and

$$\begin{aligned}
\underline{g}(x*y) &= \sup_{c\in(x*y)_\rho}\{g(c)\}\\
&= \sup_{c\in(x)_\rho(y)_\rho}\{g(c)\}\\
&= \sup_{a*b\in(x)_\rho(y)_\rho}\{g(a*b)\}\\
&\leq \sup_{b\in(y)_\rho}\{g(b)\}\\
&= \underline{g}(y) \ by\ Eq.\ (21).
\end{aligned}$$

Hence, $\rho^-(C)$ is an intuitionistic fuzzy near *UP*-filter of X.
(3) Assume that C is an intuitionistic fuzzy *UP*-filter of X and $(0)_\rho = \{0\}$. Then for all $x, y \in X$,

$$\underline{f}(0) = \inf_{a\in(0)_\rho}\{f(a)\} = f(0) \geq f(b) \geq \inf_{b\in(x)_\rho}\{f(b)\} = \underline{f}(x),$$

$$\underline{g}(0) = \sup_{a\in(0)_\rho}\{g(a)\} = g(0) \leq g(b) \leq \sup_{b\in(x)_\rho}\{g(b)\} = \underline{g}(x),$$

$$\begin{aligned}
\underline{f}(y) &= \inf_{b\in(y)_\rho}\{f(b)\}\\
&\geq \inf_{a*b\in(x)_\rho(y)_\rho, a\in(x)_\rho}\{\min\{f(a*b), f(a)\}\} \ by\ Eq.\ (23)\\
&\geq \inf_{a*b\in(x*y)_\rho, a\in(x)_\rho}\{\min\{f(a*b), f(a)\}\}\\
&= \min\{\inf_{a*b\in(x*y)_\rho}\{f(a*b)\}, \inf_{a\in(x)_\rho}\{f(a)\}\\
&= \min\{\underline{f}(x*y), \underline{f}(x)\},
\end{aligned}$$

and

$$\begin{aligned}
\underline{g}(y) &= \sup_{b\in(y)_\rho}\{g(b)\}\\
&\leq \sup_{a*b\in(x)_\rho(y)_\rho, a\in(x)_\rho}\{\max\{g(a*b), g(a)\}\} \ by\ Eq.\ (23)\\
&\leq \sup_{a*b\in(x*y)_\rho, a\in(x)_\rho}\{\max\{g(a*b), g(a)\}\}\\
&= \max\{\sup_{a*b\in(x*y)_\rho}\{g(a*b)\}, \sup_{a\in(x)_\rho}\{g(a)\}\\
&= \max\{\underline{g}(x*y), \underline{g}(x)\}.
\end{aligned}$$

Hence, $\rho^-(C)$ is an intuitionistic fuzzy *UP*-filter of X.
(4). Assume that C is an intuitionistic fuzzy *UP*-ideal of X, ρ is complete, and $(0)_\rho = \{0\}$. Then for all $x, y, z \in X$,

$$\underline{f}(0) = \inf_{a\in(0)_\rho}\{f(a)\} = f(0) \geq f(b) \geq \inf_{b\in(x)_\rho}\{f(b)\} = \underline{f}(x),$$

$$\underline{g}(0) = \sup_{a\in(0)_\rho}\{g(a)\} = g(0) \leq g(b) \leq \sup_{b\in(x)_\rho}\{g(b)\} = \underline{g}(x).$$

$$\begin{aligned}
\underline{f}(x*z) &= \inf_{d\in(x*z)_\rho}\{f(d)\} \\
&= \inf_{d\in(x)_\rho(z)_\rho}\{f(d)\} \\
&= \inf_{a*c\in(x)_\rho(y)_\rho}\{f(a*c)\} \\
&\geq \inf_{a*(b*c)\in(x)_\rho((y)_\rho(z)_\rho),b\in(y)_\rho}\{\min\{f(a*(b*c)),f(b)\}\}\ by\ Eq.\ (24) \\
&= \inf_{a*(b*c)\in(x*(y*z))_\rho,b\in(y)_\rho}\{\min\{f(a*(b*c)),f(b)\}\} \\
&= \min\{\inf_{a*(b*c)\in(x*(y*z))_\rho}\{f(a*(b*c))\},\inf_{b\in(y)_\rho}\{f(b)\} \\
&= \min\{\underline{f}(x*(y*z)),\underline{f}(y)\},
\end{aligned}$$

and

$$\begin{aligned}
\underline{g}(x*z) &= \sup_{d\in(x*z)_\rho}\{g(d)\} \\
&= \sup_{d\in(x)_\rho(z)_\rho}\{g(d)\} \\
&= \sup_{a*c\in(x)_\rho(y)_\rho}\{g(a*c)\} \\
&\leq \sup_{a*(b*c)\in(x)_\rho((y)_\rho(z)_\rho),b\in(y)_\rho}\{\min\{g(a*(b*c)),g(b)\}\}\ by\ Eq.\ (24) \\
&= \sup_{a*(b*c)\in(x*(y*z))_\rho,b\in(y)_\rho}\{\min\{g(a*(b*c)),g(b)\}\} \\
&= \min\{\sup_{a*(b*c)\in(x*(y*z))_\rho}\{g(a*(b*c))\},\sup_{b\in(y)_\rho}\{g(b)\} \\
&= \min\{\underline{g}(x*(y*z)),\underline{g}(y)\}.
\end{aligned}$$

Hence, $\rho^-(C)$ is an intuitionistic fuzzy *UP*-ideal of X.
(5). Assume that C is an intuitionistic fuzzy strong UP-ideal of U. By Theorem 2.3, C is constant. Then for all $x,y,z\in X$,

$$\underline{f}(0) = \inf_{b\in(0)_\rho}\{f(a)\} = \inf_{b\in(x)_\rho}\{f(b)\} = \underline{f}(x),$$

$$\underline{g}(0) = \sup_{b\in(0)_\rho}\{g(a)\} = \sup_{b\in(x)_\rho}\{g(b)\} = \underline{g}(x),$$

$$\begin{aligned}
\underline{f}(x) &= \inf_{a\in(x)_\rho}\{f(a)\} \\
&\geq \inf_{(c*b)*(c*a)\in((z)_\rho(x)_\rho),b\in(y)_\rho}\{\min\{f((c*b)*(c*a)),f(b)\}\}\ by\ Eq.\ (25) \\
&\geq \inf_{(c*b)*(c*a)\in((z)_\rho(x)_\rho),b\in(y)_\rho}\{\min\{f((c*b)*(c*a)),f(b)\}\} \\
&= \min\{\inf_{(c*b)*(c*a)\in((z*x))_\rho,b\in(y)_\rho}\{f((c*b)*(c*a))\},\inf_{b\in(y)_\rho}\{f(b)\} \\
&= \min\{\underline{f}((z*y)*(z*x)),\underline{f}(y)\},
\end{aligned}$$

and

$$\begin{aligned}
\underline{g}(x) &= \sup_{a\in(x)_\rho}\{g(a)\} \\
&\leq \sup_{(c*b)*(c*a)\in((z)_\rho(x)_\rho),b\in(y)_\rho}\{\max\{g((c*b)*(c*a)),g(b)\}\}\ by\ Eq.\ (25) \\
&\leq \sup_{(c*b)*(c*a)\in((z)_\rho(x)_\rho),b\in(y)_\rho}\{\max\{g((c*b)*(c*a)),g(b)\}\} \\
&= \max\{\sup_{(c*b)*(c*a)\in((z*x))_\rho,b\in(y)_\rho}\{g((c*b)*(c*a))\},\sup_{b\in(y)_\rho}\{g(b)\} \\
&= \max\{\underline{f}((z*y)*(z*x)),\underline{g}(y)\}.
\end{aligned}$$

Hence, $\rho^-(C)$ is an intuitionistic fuzzy strong *UP*-ideal of X. $\square$

The following example shows that Theorem 3.4 (3) may be not true if $(0)_\rho \neq \{0\}$.

Example 3.1. Let $X = \{0, 1, 2, 3\}$ be a *UP*-algebra with a fixed element 0 and a binary operation $*$ defined by the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	0
2	0	1	0	3
3	0	1	2	0

An intuitionistic fuzzy set $C = (f, g)$ with f and g is defined as follows:
Then, $C = (f, g)$ is an intuitionistic fuzzy *UP*-filter of X.

C	0	1	2	3
f	0.71	0.42	0.63	0.63
g	0.21	0.63	0.34	0.34

Let $\rho = \{(0,0),(1,1),(2,2),(3,3),(0,1),(1,0),(0,3),(3,0)\}$. Then, ρ is a congruence relation on X. Thus, $(0)_\rho = (1)_\rho = (3)_\rho = \{0,1,3\}$, $(2)_\rho = \{2\}$.
Since $\underline{f}(0) = min\{f(0), f(1), f(3)\} = \min\{0.71, 0.42, 0.63\} = 0.42 \ngeq 0.63 = f(2) = \underline{f}(2)$ and $\underline{g}(0) = max\{g(0), g(1), g(3)\} = max\{0.21, 0.63, 0.34\} = 0.63 \nleq 0.34 = g(2) = \underline{g}(2)$, $\rho^-(C)$ is not an intuitionistic fuzzy *UP*-filter of X.

The following example shows that Theorem 3.4 (4) may be not true if $(0)_\rho \neq \{0\}$ and ρ is not complete.

Example 3.2. From Example 3.1, an intuitionistic fuzzy set $C = (f, g)$ with f and g is defined as follows:

C	0	1	2	3
f	1	0.22	0.13	0.54
g	0	0.63	0.94	0.45

Then, $C = (f, g)$ is an intuitionistic fuzzy *UP*-ideal of X.
Let $\rho = \{(0,0),(1,1),(2,2),(3,3),(0,2),(2,0)\}$. Then, ρ is a congruence relation on X. Thus, $(0)_\rho = (2)_\rho = \{0,2\}$, $(1)_\rho = \{1\}$, $(3)_\rho = \{3\}$. Since $\underline{f}(0) = min\{f(0), f(2)\} = min\{1, 0.13\} = 0.13 \ngeq 0.22 = f(1) = \underline{f}(1)$ and $\underline{g}(0) = max\{g(0), g(2)\} = max\{0, 0.94\} = 0.94 \nleq 0.63 = g(1) = \underline{g}(1)$, $\rho^-(C)$ is not an intuitionistic fuzzy *UP*-ideal of X.

Theorem 3.5. *If ρ is an congruence relation on a UP-algebra $X = (X, *, 0)$ and $C = (f, g)$ a intuitionistic fuzzy UP-subalgebra of X, then the upper approximation $\rho^+(C)$ satisfies the following conditions:*

$$\{\forall x \in X\}, \left(x\rho y \Rightarrow \left\{ \begin{array}{l} \overline{f}(0) \geq \overline{f}(x), \\ \overline{g}(0) \leq \overline{g}(x). \end{array} \right. \right). \tag{27}$$

Proof. Let $x \in X$. Then,

$$\overline{f}(0) = \sup_{a\in(x)_\rho}\{f(a)\} \geq f(0) \geq \sup_{a\in(x)_\rho}\{f(a)\} = \overline{f}(x) \text{ by } Eq. (22)$$

and

$$\overline{g}(0) = \inf_{a\in(x)_\rho}\{g(a)\} \leq g(0) \leq \inf_{a\in(x)_\rho}\{g(a)\} = \overline{g}(x) \text{ by } Eq. (22).$$

Hence, $\rho^+(C)$ satisfies Eq. (27). □

Theorem 3.6. *Let ρ be an congruence relation on a UP-algebra $X = (X, *, 0)$ and $C = (f, g)$ an intuitionistic fuzzy set in X. Then, the following statements hold:*

(1) If C is an intuitionistic fuzzy UP-subalgebra of X, then $\rho^+(C)$ is an intuitionistic fuzzy UP-subalgebra of X,
(2) If C is an intuitionistic fuzzy near UP-filter of X, then $\rho^+(C)$ is an intuitionistic fuzzy near UP-filter of X, and
(3) If C is an intuitionistic fuzzy strong UP-ideal of X, then $\rho^+(C)$ is an intuitionistic fuzzy strong UPideal of X.

Proof. (1) Assume that C is an intuitionistic fuzzy *UP*-subalgebra of X. Then for all $x, y \in X$, Case 1: $x = y$. Then

$$\overline{f}(x * y) = \overline{f}(0) \geq \overline{f}(x) \geq \min\{\overline{f}(x), \overline{f}(y)\} \text{ by } Eqs. (7) \text{ and } (27)$$

and

$$\overline{g}(x * y) = \overline{g}(0) \leq \overline{g}(x) \leq \max\{\overline{g}(x), \overline{g}(y)\} \text{ by } Eqs. (7) \text{ and } (27).$$

Case 2: $x \neq y$. Case 2.1: $x * y = x$ or y. It is sufficient to assume that $x * y = x$. Then

$$\overline{f}(x * y) = \overline{f}(x) \geq \min\{\overline{f}(x), \overline{f}(y)\}$$

and

$$\overline{g}(x * y) = \overline{g}(x) \leq \max\{\overline{g}(x), \overline{g}(y)\}.$$

Case 2.2: $x * y \neq x$ and $x * y \neq y$. Assume that there exists $z \in X$ be such that $x * y = z$. If $z\rho0$, then

$$\overline{f}(x * y) = \overline{f}(z) = \overline{f}(0) \geq \min\{\overline{f}(x), \overline{f}(y)\} \text{ by } Eq. (26)$$

and

$$\overline{g}(x * y) = \overline{g}(z) = \overline{g}(0) \leq \max\{\overline{g}(x), \overline{g}(y)\} \text{ by } Eq. (26).$$

If $x\rho0$ or $y\rho0$, it is sufficient to assume that $x\rho0$. Since ρ is a congruence relation on X, we have $xy\rho0y$, that is, $z\rho y$. Therefore,

$$f(x * y) = f(z) = f(y) = \min\{f(0), f(y)\} = \min\{f(x), f(y)\} \text{ by } Eqs. (26) \text{ and } (27)$$

and

$$g(x * y) = g(z) = g(y) = \min\{g(0), g(y)\} = \max\{g(x), g(y)\} \text{ by } Eqs. (26) \text{ and } (27).$$

Hence, $\rho^+(C)$ is an intuitionistic fuzzy *UP*-subalgebra of X.

(2). Assume that C is an intuitionistic fuzzy near *UP*-filter of X. Then for all $x, y \in X$,

$$\begin{aligned}\overline{f}(x*y) &= \sup_{c\in(x*y)_\rho}\{f(c)\}\\ &\geq \sup_{c\in(x)_\rho(y)_\rho}\{f(c)\}\\ &= \sup_{a*b\in(x)_\rho(y)_\rho}\{f(a*b)\}\\ &\geq \sup_{b\in(y)_\rho}\{f(b)\}\\ &= \overline{f}(y) \ by\ Eq.\ (21)\end{aligned}$$

and

$$\begin{aligned}\overline{g}(x*y) &= \inf_{c\in(x*y)_\rho}\{g(c)\}\\ &\leq \inf_{c\in(x)_\rho(y)_\rho}\{g(c)\}\\ &= \inf_{a*b\in(x)_\rho(y)_\rho}\{g(a*b)\}\\ &\leq \inf_{b\in(y)_\rho}\{g(b)\}\\ &= \overline{g}(y) \ by\ Eq.\ (21).\end{aligned}$$

Hence, $\rho^+(C)$ is an intuitionistic fuzzy near *UP*-filter of X.

(3) Assume that C is an intuitionistic fuzzy strong *UP*-ideal of U. By Theorem 2.3, C is constant. Then for all $x, y, z \in X$,

$$\overline{f}(0) = \sup_{a\in(0)_\rho}\{f(a)\} = \sup_{b\in(x)_\rho}\{f(b)\} = \overline{f}(x),$$

$$\overline{g}(0) = \inf_{b\in(0)_\rho}\{g(a)\} = \inf_{b\in(x)_\rho}\{g(b)\} = \overline{g}(x).$$

$$\begin{aligned}\overline{f}(x) &= \sup_{a\in(x)_\rho}\{f(a)\}\\ &\geq \sup_{(c*b)*(c*a)\in((z)_\rho(x))_\rho),b\in(y)_\rho}\{\min\{f((c*b)*(c*a)), f(b)\}\} \ by\ Eq.\ (25)\\ &\geq \sup_{(c*b)*(c*a)\in((z)_\rho(x)_\rho),b\in(y)_\rho}\{\min\{f((c*b)*(c*a)), f(b)\}\}\\ &= \min\{\sup_{(c*b)*(c*a)\in((z*x))_\rho,b\in(y)_\rho}\{f((c*b)*(c*a))\}, \sup_{b\in(y)_\rho}\{f(b)\}\\ &= \min\{\overline{f}((z*y)*(z*x)), \overline{f}(y)\},\end{aligned}$$

and

$$\begin{aligned}\overline{g}(x) &= \inf_{a\in(x)_\rho}\{g(a)\}\\ &\leq \inf_{(c*b)*(c*a)\in((z)_\rho(x))_\rho),b\in(y)_\rho}\{\max\{g((c*b)*(c*a)), g(b)\}\} \ by\ Eq.\ (25)\\ &\leq \inf_{(c*b)*(c*a)\in((z)_\rho(x)_\rho),b\in(y)_\rho}\{\max\{g((c*b)*(c*a)), g(b)\}\}\\ &= \max\{\inf_{(c*b)*(c*a)\in((z*x))_\rho,b\in(y)_\rho}\{g((c*b)*(c*a))\}, \inf_{b\in(y)_\rho}\{g(b)\}\\ &= \max\{\overline{g}((z*y)*(z*x)), \overline{g}(y)\}.\end{aligned}$$

Hence $\rho^+(C)$ is an intuitionistic fuzzy strong *UP*-ideal of X. □

The following example shows that if C is an intuitionistic fuzzy *UP*-filter of U, then the upper approximation $\rho^+(C)$ is not an intuitionistic fuzzy *UP*-filter in general.

Example 3.3. Let $X = \{0, 1, 2, 3\}$ be a *UP*-algebra with a fixed element 0 and a binary operation $*$ defined by the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

An intuitionistic fuzzy set $C = (f, g)$ with f and g is derived as follows:

C	0	1	2	3
f	0.61	0.52	0.33	0.33
g	0.31	0.42	0.73	0.73

Then, $C = (f, g)$ is an intuitionistic fuzzy *UP*-filter of X.

Let $\rho = \{(0,0), (1,1), (2,2), (3,3), (3,0), (0,3)\}$. Then ρ is a congruence relation on X. Thus $(0)_\rho = (3)_\rho = \{0, 3\}$, $(1)_\rho = \{1\}$, $(2)_\rho = \{2\}$. Since $\overline{f}(2) = f(2) = 0.33 \ngeq 0.52 = \min\{\max\{f(0), f(3)\}, f(1)\} = \min\{\underline{f}(3), \underline{f}(1)\} = \min\{\underline{f}(1 * 2), \underline{f}(1)\}$. We have $\rho^+(C)$ is not an intuitionistic fuzzy *UP*-filter of X.

4. Conclusions and future works

In this paper, the concept of intuitionistic fuzzy sets in *UP*-algebras is introduced, and five types of intuitionistic fuzzy sets in *UP*-algebras, namely intuitionistic fuzzy *UP*-subalgebras, intuitionistic fuzzy near *UP*-filters, intuitionistic fuzzy *UP*-filters, intuitionistic fuzzy *UP*-ideals, and intuitionistic fuzzy strong *UP*-ideals were introduced. Further, the relationship between some assertions of intuitionistic fuzzy sets and intuitionistic fuzzy *UP*-subalgebras (resp., intuitionistic fuzzy near *UP*-filters, intuitionistic fuzzy *UP*-filters, intuitionistic fuzzy *UP*-ideals, intuitionistic fuzzy strong *UP*-ideals) in *UP*-algebras also discussed and upper and lower approximations of intuitionistic fuzzy sets is studied.

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