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Confidence Interval for Parameters Estimates in Circular Simultaneous Functional Relationship Model (CSFRM) for Equal Variances using Normal Asymptotic and Bootstrap Confidence Intervals

(Selang Keyakinan Anggaran Parameter untuk Model Hubungan Fungsian Membulat Serentak (CSFRM) dengan Andaian Ralat Varians sama menggunakan Pendekatan Asimptot dan Pembutstrapan)

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ABSTRACT

Few studies have considered the functional relationship model for circular variables. Anuar has proposed a new model of Circular Simultaneous Functional Relationship Model for equal variances. However, the confidence interval for all parameter estimates in this model has not received any consideration in any literature. This paper proposes the confidence interval for all parameter estimates of von Mises distribution in this model. The parameters are estimated using minimum sum (ms) and polyroot function provided in (built-in package) Splus statistical software. The parameters confidence may be obtained from parameter estimation. Those estimation values are obtained by minimizing the negative value of the log-likelihood function. Then, the confidence interval for all parameters based on the bootstrap method will be compared with the normal asymptotic confidence interval via simulation studies. It is found that bootstrap method is the superior method by measuring the performance using coverage probability and expected length. The confidence intervals are illustrated using real wind direction data of Bayan Lepas that collected at 16.3 m above ground level, latitude 05°18’N and longitude 100°16’E. The results showed that the estimate parameters fall between the estimate interval, and we note that the method works well for this model.

Keywords: Bootstrap confidence interval; circular simultaneous functional relationship model; normal asymptotic confidence interval; parameters estimate; von Mises distribution

ABSTRAK

Beberapa kajian telah mempertimbangkan model hubungan fungsian untuk pemboleh ubah membulat. Anuar telah mencadangkan model baru iaitu Model Hubungan Fungsian Membulat Serentak dengan Andaian Ralat Varians Sama. Walau bagaimanapun, selang keyakinan semua anggaran parameter untuk model ini tidak mendapat pertimbangan di mana-mana kepustakaan. Kajian ini mencadangkan selang keyakinan untuk semua anggaran parameter taburan von Mises dalam model ini. Parameter dianggarkan menggunakan fungsi minimum sum (ms) dan fungsi polyroot yang dibekalkan (*built-in*) dalam perisian statistik Splus. Keyakinan parameter boleh didapati daripada anggaran parameter. Nilai anggaran tersebut boleh diperoleh dengan meminimumkan nilai negatif fungsi kemungkinan log. Kemudian, selang keyakinan terhadap semua anggaran parameter berdasarkan kaedah pembustrapan dibandingkan dengan kaedah normal asimptot melalui kajian simulasi. Didapati kaedah pembustrapan adalah kaedah unggul dengan mengukur prestasi menggunakan kebarangkalian liputan dan jangkaan panjang. Kaedah ini diilustrasikan menggunakan data arah angin Bayan Lepas yang dikumpul pada 16.3 m di atas paras tanah, latitud 05°18’N dan longitud 100°16’E. Hasil kajian menunjukkan bahawa semua anggaran parameter jatuh antara selang anggaran dan kaedah tersebut berfungsi dengan baik untuk model ini.

Kata kunci: Anggaran parameter; model hubungan fungsian membulat serentak; selang keyakinan pembutstrapan; selang keyakinan normal asimptot; taburan von Mises

INTRODUCTION

Circular data or directional data has its own distributional topologies. Circular data refer to a set of observations measured in degree $\left(0^o, 360^o\right]$ or radian $\left(0, 2\pi\right]$. It can be shown on the circumference of a unit circle. The analysis of circular variables or directional data has gained substantial attention in recent years as several phenomenon in real-life applications have been detected. In 2020, the directional data has been applied by Ahmad et al. to study a new crescent moon visibility criterion. Directional data also can be found in many scientific fields for instance in movement coordination (Stock et al. 2018), physics and biology (Fitak, Caves & Johnsen 2018), meteorology, medicine, and geology. In meteorology, the applications of circular statistics have been used by meteorologists to study the rate of heavy rain in a year (Mardia & Jupp 2000) and wind directions (Gatto & Jammalamadaka 2007). Next in medicine, the application of medical has been discussed by Jammalamadaka et al. (1986) that the recovery of orthopaedic patients can be accessed by measuring the angle of knee flexion. While geologist consider the directional data used in modelling cross-bedding data (Jones & James 1969) and earthquake displacement direction (Rivest 1997).

The theories and the statistical methods of circular data are developed over the years (Badarisam, Rambli & Sidik 2020; Hussin et al. 2013; Mohamed et al. 2016; Rambli et al. 2015, 2012). However, they can be further improved and refined in many statistical aspects. The first statistical review paper regarding the development of circular data analysis was from Jupp and Mardia (1989). As for the circular data case, Hassan (2010) stated that due to the complexity of the circular data, there are limited references to functional relationship for circular variables. Satari et al. (2014) have developed the functional relationship model for circular variables to study the error in variables (EIV) for Down Mardia (DM) circular regression model. The Circular Functional Relationship Model (CFRM) proposed by Satari et al. (2014) is an extension of the DM circular regression model to the error in variables model and it involves the study of the relationship between two circular variables. In 2018, Anuar proposed the Circular Simultaneous Functional Relationship Model (CSFRM) for equal error variances. Anuar's model is an extension model from Satari et al. (2014)'s model, where the model involves more than one circular dependent variable $\left(y_1, y_2\right)$, simultaneously with a circular independent variable, $\left(x_1\right)$. Von Mises distribution is commonly used as a distribution to describe the circular random variable. The probability density function (*pdf*) is given by

$$f\left(\theta;\mu,\kappa\right)=\frac{1}{2\pi I_o\left(\kappa\right)}\exp\left(\kappa\cos\left(\theta-\mu\right)\right),\ \ 0\le\theta<2\pi,\kappa\ge0 \quad (1)$$

where μ is the mean direction and κ is the concentration parameter. $I_o\left(\kappa\right)$ is the modified Bessel function of the first kind and order zero and it can be clarified as

$$I_o\left(\kappa\right)=\frac{1}{2\pi}\int_0^{2\pi}e^{\kappa\cos\left(\theta-\mu\right)}d\theta \quad (2)$$

This is a continuous probability distribution, and the distribution will be converged to the uniform distribution as κ approaches 0. Meanwhile, as κ increase, the uniform distribution converges to the point distribution and it is concentrated in the direction μ. Hence, Fisher (1993) and Mardia (1972) stated that it will approach the normal distribution with the mean μ_0 and variance. Besides, the concentration parameter, κ approximation to the normal distribution has also been tested by Moslim et al. (2017) via a simulation study.

Confidence interval is often used in data analysis, and it is known as the interval for the estimated point where the true value of the parameter lies within the interval. Based on reviewed literature, the confidence interval for parameters estimates in Circular Simultaneous Functional Relationship Model (CSFRM) for equal variances has not been published in any literatures. Therefore, in this paper, estimation of confidence interval is proposed for all parameters of von Mises distribution using Normal Asymptotic Confidence Interval (NACI) and Bootstrap Confidence Interval (BCI) in CSFRM.

The angular parameters and slope parameter, $\alpha_1, \alpha_2, \beta_1, \beta_2, \omega_1, \omega_2$ are estimated using minimum sum (ms) method and concentration parameter, κ is estimated using polyroot method. Since, the close form of estimators cannot be obtained, the minimum sum (ms) function is used to estimate the parameters and concentration parameter, κ is estimated using polyroot method. The simulation studies are conducted to access the accuracy of the CSFRM for equal variances and compared with both methods of NACI and BCI. Their performance is measured using coverage probability and expected length (Hassan, Zubairi & Hussin 2012; Hassan et al. 2014; Letson & McCullough 1998). Then, the proposed confidence interval for all parameters estimates is illustrated by using real wind direction data set.

ESTIMATION OF PARAMETER USING ms AND POLYROOT FUNCTION

The method for parameters' estimation used in this study are minimum sum (ms) and polyroot function. The estimation of parameters is obtained directly by applying those functions which were provided in the circular statistics library of Splus statistical package. The estimation values of parameters are estimated by minimizing the negative value of the log-likelihood function. Thus, to find the estimators for angular and slope parameters, the ms function is applied to log-likelihood function equations with a set of initial values which correspond to the maximum value of the precision parameter. Meanwhile, for polyroot function is based on the power series expansion of the mean resultant length and the estimation of concentration parameters is obtained from roots of polynomial function.

CONFIDENCE INTERVAL BASED ON ASYMPTOTIC DISTRIBUTION (NACI)

The confidence interval for all parameters is constructed based on the variance-covariance matrix derived from Fisher information matrix. Therefore, the Fisher information matrix for the distribution of $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\omega}_1, \hat{\omega}_2$ is defined by

$$\hat{\alpha}_1, \hat{\beta}_1, \hat{\omega}_1 \sim N\left(\left(\hat{\alpha}_1, \hat{\beta}_1, \hat{\omega}_1\right), \frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_1, \hat{\beta}_1, \hat{\omega}_1\right)} \right), \quad (3)$$

$$\hat{\alpha}_2, \hat{\beta}_2, \hat{\omega}_2 \sim N\left(\left(\hat{\alpha}_2, \hat{\beta}_2, \hat{\omega}_2\right), \frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_2, \hat{\beta}_2, \hat{\omega}_2\right)} \right). \quad (4)$$

The 100 (1-α)% confidence interval for $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\omega}_1, \hat{\omega}_2$ is given by

$$\hat{\alpha}_1 - \frac{z_{\alpha/2}}{\left[\frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_1, \hat{\beta}_1, \hat{\omega}_1\right)}\right]^{1/2}}, \hat{\alpha}_1 + \frac{z_{\alpha/2}}{\left[\frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_1, \hat{\beta}_1, \hat{\omega}_1\right)}\right]^{1/2}} \quad (5)$$

$$\hat{\alpha}_2 - \frac{z_{\alpha/2}}{\left[\frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_2, \hat{\beta}_2, \hat{\omega}_2\right)}\right]^{1/2}}, \hat{\alpha}_2 + \frac{z_{\alpha/2}}{\left[\frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_2, \hat{\beta}_2, \hat{\omega}_2\right)}\right]^{1/2}} \quad (6)$$

$$\hat{\beta}_1 - \frac{z_{\alpha/2}}{\left[\frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_1, \hat{\beta}_1, \hat{\omega}_1\right)}\right]^{1/2}}, \hat{\beta}_1 + \frac{z_{\alpha/2}}{\left[\frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_1, \hat{\beta}_1, \hat{\omega}_1\right)}\right]^{1/2}} \quad (7)$$

$$\hat{\beta}_2 - \frac{z_{\alpha/2}}{\left[\frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_2, \hat{\beta}_2, \hat{\omega}_2\right)}\right]^{1/2}}, \hat{\beta}_2 + \frac{z_{\alpha/2}}{\left[\frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_2, \hat{\beta}_2, \hat{\omega}_2\right)}\right]^{1/2}} \quad (8)$$

$$\hat{\omega}_1 - \frac{z_{\alpha/2}}{\left[\frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_1, \hat{\beta}_1, \hat{\omega}_1\right)}\right]^{1/2}}, \hat{\omega}_1 + \frac{z_{\alpha/2}}{\left[\frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_1, \hat{\beta}_1, \hat{\omega}_1\right)}\right]^{1/2}} \quad (9)$$

$$\hat{\omega}_2 - \frac{z_{\alpha/2}}{\left[\frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_2, \hat{\beta}_2, \hat{\omega}_2\right)}\right]^{1/2}}, \hat{\omega}_2 + \frac{z_{\alpha/2}}{\left[\frac{1}{\tilde{\kappa} A(\tilde{\kappa}) B\left(\hat{\alpha}_2, \hat{\beta}_2, \hat{\omega}_2\right)}\right]^{1/2}} \quad (10)$$

The 100 (1-α)% confidence interval for κ is defined by

$$\left(\hat{k} - \frac{z_{\alpha/2}}{\left[2n\left(1 - \frac{A(\hat{\kappa})}{\hat{\kappa}} - A^2(\hat{\kappa})\right)\right]^{\frac{1}{2}}}, \hat{\kappa} + \frac{z_{\alpha/2}}{\left[2n\left(1 - \frac{A(\hat{\kappa})}{\hat{\kappa}}\right) - A^2(\hat{\kappa})\right]^{\frac{1}{2}}} \right). \quad (11)$$

where

$$A(\hat{\kappa}) = \begin{cases} A_s(\hat{\kappa}) = \frac{\hat{\kappa}}{2} - \frac{\hat{\kappa}^3}{16} + \frac{\hat{\kappa}^5}{96} & \hat{\rho} < 0.613637, \\ A_L(\hat{\kappa}) = 1 - \frac{1}{2\hat{\kappa}} - \frac{1}{8\hat{\kappa}^2} + \frac{1}{8\hat{\kappa}^3} & \hat{\rho} \geq 0.613637. \end{cases} \quad (12)$$

CONFIDENCE INTERVAL OF $\alpha_1, \alpha_2, \beta_1, \beta_2, \omega_1, \omega_2$ BASED ON BOOTSTRAP METHOD (BCI)

The following steps describe bootstrap method for $\alpha_1, \alpha_2, \beta_1, \beta_2, \omega_1, \omega_2$:

Step 1: Re-sampling. The sample size of $m < n$ is drawn with replacement from the original sample. Next, the usual estimate of $\alpha_1, \alpha_2, \beta_1, \beta_2, \omega_1, \omega_2$ are the $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\omega}_1, \hat{\omega}_2$ and κ is the $\tilde{\kappa}$ for CFRM. Then, simulate values m of the observe variables x and y using CSFRM with fixed values of $\alpha_1, \alpha_2, \beta_1, \beta_2, \omega_1, \omega_2$ and $\tilde{\kappa}$ based from the generated random samples X, δ and ε.

Step 2: Estimate $\tilde{\kappa}_1^*$ with m values of the observed variables x and y from CSFRM.

Step 3: Repeat step 1 and step 2 to obtain bootstrap, B=200. The estimate is $\alpha_1, ..., \alpha_B$, $\alpha_2, ..., \alpha_B, \beta_1, ..., \beta_B, \beta_2, ..., \beta_B, \omega_1, ...\omega_B$, and $\omega_2, ...,$ $...., \omega_B$.

Step 4: Then, sort the bootstrap estimate (Step 3) in ascending order to obtain $\hat{\alpha}_1^* \leq .... \leq \hat{\alpha}_B^*, \hat{\alpha}_2^* \leq .... \leq \hat{\alpha}_B^*, \hat{\beta}_1^* \leq .... \leq \hat{\beta}_B^*, \hat{\beta}_2^* \leq .... \leq \hat{\beta}_B^*$, $\hat{\omega}_1^* \leq .... \leq \hat{\omega}_B^*$, and $\hat{\omega}_2^* \leq .... \leq \hat{\omega}_B^*$.

The 100 (1-α)% confidence interval for $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\omega}_1, \hat{\omega}_2$ is defined by

$$\left(\hat{\alpha}^*_{(l+1)}, \hat{\alpha}^*_{(m)}\right) \quad (13)$$

$$\left(\hat{\beta}^*_{(l+1)}, \hat{\beta}^*_{(m)}\right) \quad (14)$$

$$\left(\hat{\omega}^*_{(l+1)}, \hat{\omega}^*_{(m)}\right) \quad (15)$$

where l = integer part of $\left(\frac{1}{2}B\alpha + \frac{1}{2}\right)$ and $m = B - l$ and $\alpha = 0.05$.

CONFIDENCE INTERVAL OF κ BASED ON BOOTSTRAP METHOD (BCI)

The following steps describe bootstrap method for κ:
Step 1: Re-sampling. Draw a random sample of size $m<n$ with replacement from the original sample. Let $\hat{\kappa}$ be the usual estimate of κ for CFRM. Then, simulate m values of the observe variables x and y using CFRM with fixed values of $\hat{\kappa}$ based from the generated random samples X, δ and ε.
Step 2: Estimate $\tilde{\kappa}^*_1$ with m values of the observed variables x and y from CFRM.
Step 3: Repeat step 1 and step 2 for obtaining a total of B = 200, where B is the number of bootstrap estimates.
Step 4: After the total of B estimates is obtained, the bootstrap estimate $\tilde{}^*_1,\ldots,\tilde{}^*_B$, need to be sorted in ascending order to obtain $\tilde{\kappa}^*_1 \leq \ldots \leq \tilde{\kappa}^*_B$. The 100 (1-)% confidence interval for is defined by

$$\left(\tilde{\kappa}^*_{(l+1)}, \tilde{\kappa}^*_{(m)}\right). \quad (16)$$

where l = integer part of $\left(\frac{1}{2}B\alpha + \frac{1}{2}\right)$ and $m = B - l$ and $\alpha = 0.05$.

SIMULATION STUDIES

Simulation study is done using SPlus statistical package with different random samples size of n = 30,50,100 and set of concentration parameter, κ = 10,15,20. Then, the values of the observe variables x and y_1, y_2 are calculated using the new circular simultaneous functional relationship model for equal variance with fixed values of $\alpha_1 = 0.785$, $\alpha_2 = 0.524$, $\beta_1 = 0.785$, $\beta_2 = 0.524$ and for both ω is fixed values of $\omega = 0.5$. Afterwards, the confidence intervals for all parameters are obtained at confidence level $\alpha = 0.05$ based on methods NACI and BCI. The efficiency of both methods NACI and BCI is measured by comparing the coverage probability and expected length. The simulation study is repeated 000 times and the coverage probability (CP) is calculated as below

$$\text{Coverage Probability} = \frac{q}{s} \quad (17)$$

where q is the number of true value of falls in the confidence interval (CI); and s is the number of simulations.

The expected length is calculated as follows

$$\text{Expected length} = \text{Upper limit} - \text{Lower limit} \quad (18)$$

Hassan et al. (2012) stated that the coverage probability is the proportion of the time that the confidence interval contains the true value of error concentration parameter. Meanwhile, the expected length is the size of the confidence interval. The best method of confidence interval will have the closest coverage probability to the nominal coverage probability of 0.95 and the shortest expected length for 95% confidence interval (Moslim et al. 2019).

RESULTS AND DISCUSSIONS

Tables 1 to 4 summarize the performance of coverage probability and expected length for angular parameter, slope parameter and concentration parameter, κ and samples sizes, n for both methods. Both methods and performance measures are labelled as below:

a. Performance measures:

   i. CP 95% - coverage probability

   ii. EL - expected length

a. Methods:

   i. NACI - normal asymptotic confidence interval

   ii. BCI – bootstrap confidence interval

TABLE 1. Coverage probability and expected length for angular parameter, α_1 and α_2

		α_1				α_2			
		CP 95%		EL		CP 95%		EL	
κ	n	NACI	BCI	NACI	BCI	NACI	BCI	NACI	BCI
	30	0.765	0.895	3.881	4.514	0.767	0.831	2.122	3.509
10	50	0.894	0.900	4.143	3.009	0.864	0.888	2.780	3.051
	100	0.902	0.932	4.569	2.941	0.891	0.901	3.533	2.734
	30	0.800	0.905	4.321	3.377	0.874	0.881	3.098	2.819
15	50	0.921	0.925	4.914	2.349	0.925	0.938	3.137	2.786
	100	0.945	0.945	5.001	1.890	0.933	0.946	3.877	2.558
	30	0.899	0.937	4.447	2.090	0.931	0.911	4.122	2.463
20	50	0.931	0.942	5.390	1.773	0.940	0.947	4.394	2.312
	100	0.940	0.955	5.989	1.116	0.949	0.955	4.501	1.927

The simulation results for confidence interval estimation of angular parameter α_1 and α_2 are shown in Table 1. From Table 1, the coverage probability for bootstrap method (BCI) always closer to the nominal coverage probability of 0.95 and reach the exact value. In comparison, the asymptotic method (NACI) closes to the nominal coverage probability of 0.95 but do not reach the exact value. Therefore, bootstrap method is the best method in finding the confidence interval for estimation of angular parameter, α_1 and α_2. Based on the results of expected length, bootstrap method seems to give the smallest values, thus suggesting its superiority as compared to the asymptotic method. This may indicated that bootstrap method is good to construct the confidence interval of angular parameter α_1 and α_2.

TABLE 2. Coverage probability and expected length for angular parameter, β_1 and β_2

		β_1				β_2			
		CP 95%		EL		CP 95%		EL	
κ	n	NACI	BCI	NACI	BCI	NACI	BCI	NACI	BCI
	30	0.690	0.856	1.483	1.500	0.770	0.840	1.320	1.903
10	50	0.718	0.914	1.505	1.370	0.887	0.908	1.494	1.832
	100	0.878	0.928	1.690	1.289	0.900	0.920	1.511	1.778
	30	0.742	0.869	2.012	1.233	0.815	0.878	2.208	1.346
15	50	0.813	0.937	2.371	1.130	0.890	0.931	2.363	1.553
	100	0.899	0.941	2.579	1.001	0.908	0.944	2.478	1.230
	30	0.864	0.939	2.099	0.808	0.871	0.934	2.499	0.915
20	50	0.886	0.948	3.0005	0.740	0.909	0.955	2.573	0.780
	100	0.900	0.952	3.137	0.676	0.911	0.955	2.599	0.458

The simulation results for confidence interval estimation of angular parameter β_1 and β_2 are shown in Table 2. From Table 2, similar findings are obtained as angular parameter of α_1 and α_2. As the increase of sample size and concentration parameter, the coverage probability for bootstrap method (BCI) also increase. The BCI results closest to the nominal coverage probability of 0.95 and reach the exact value. In comparison, the asymptotic method (NACI) also closest to the nominal coverage probability of 0.95 but do not reach the exact value. Therefore, bootstrap method is the best method in finding the confidence interval for estimation of angular parameter, β_1 and β_2. Based on the results of expected length, bootstrap method seems to give the smallest values, as compared to the asymptotic method. This may indicate that bootstrap method is good to construct the confidence interval of angular parameter β_1 and β_2.

TABLE 3. Coverage probability and expected length for slope parameter, ω_1 and ω_2

		ω_1				ω_2			
		CP 95%		EL		CP 95%		EL	
κ	n	NACI	BCI	NACI	BCI	NACI	BCI	NACI	BCI
	30	0.709	0.771	1.400	0.540	0.800	0.866	1.352	1.118
10	50	0.816	0.854	1.443	0.582	0.916	0.900	1.422	0.634
	100	0.880	0.898	1.560	0.441	0.920	0.911	2.323	0.541
	30	0.874	0.879	1.636	0.376	0.879	0.899	1.478	1.012
15	50	0.891	0.899	1.744	0.391	0.924	0.913	2.115	0.455
	100	0.900	0.910	1.770	0.379	0.937	0.940	2.764	0.333
	30	0.900	0.925	1.801	0.288	0.914	0.922	1.574	1.009
20	50	0.923	0.944	1.813	0.292	0.939	0.939	2.637	0.302
	100	0.945	0.955	1.855	0.211	0.940	0.949	2.881	0.294

Next, the simulation results for confidence interval estimation of angular parameter ω_1 and ω_2 are shown in Table 3. From Table 3, similar findings are obtained as angular parameter of $\alpha_1, \alpha_2, \beta_1$ and β_2. There is an increment in coverage probability for bootstrap method (BCI) as the increase number of sample size and concentration parameter. The BCI results closest to the nominal coverage probability of 0.95 and reach the exact value. In comparison, the asymptotic method (NACI) closest to the nominal coverage probability of 0.95 but do not reach the exact value. Therefore, bootstrap method is the best method in finding the confidence interval for estimation of angular parameter, ω_1 and ω_2. Based on the results of expected length, bootstrap method seems to give the smallest values, thus suggesting its superiority as compared to the asymptotic

TABLE 4. Coverage probability and expected length for error concentration parameter, κ

		κ			
		CP 95%		EL	
κ	n	NACI	BCI	NACI	BCI
10	30	0.676	0.712	4.567	4.346
	50	0.786	0.842	5.717	5.258
	100	0.800	0.877	8.986	8.776
15	30	0.863	0.900	10.121	9.010
	50	0.881	0.911	13.419	12.758
	100	0.892	0.932	14.111	13.220
20	30	0.900	0.933	15.001	14.891
	50	0.922	0.943	15.525	15.492
	100	0.942	0.956	15.872	15.625

method. This may indicate that bootstrap method is good to construct the confidence interval of angular parameter, ω_1 and ω_2.

The simulation results shown in Table 4 is the confidence interval for error concentration parameter, κ. From Table 4, as sample size, n and concentration parameter, κ increases, the coverage probability for bootstrap method gets closer to the nominal coverage probability of 0.95. In comparison, the coverage probability for asymptotic method also gets closer to the nominal coverage probability of 0.95 but never reach the exact value. If we look solely into the coverage probability, it helps to indicate that the bootstrap method is the best method in finding the confidence interval for the error concentration parameter, κ. Based on the performance measure of the expected length, the bootstrap method seems to give the smallest values, thus suggesting its superiority as compared to the asymptotic method.

PRACTICAL EXAMPLES

As a practical example for the proposed confidence interval, Bayan Lepas data recorded at latitude 05°18’N and longitude 100°16’E were considered. These data were collected in the year of 2005 and consisted 62 observations with 5000 m height at pressure 850 Hpa (x), 300 m height at pressure 1000 Hpa (y_1) and 1900 m height at pressure 500 Hpa (y_2) as independent and dependent variables.

The results shown in Table 5 are obtained using the bootstrap confidence interval. Based on the data with 95% confidence, all angular and slope parameters always fall between the estimate intervals. Hence, it can be concluded that the proposed confidence interval for all parameters are works well for Circular Simultaneous Functional Relationship Model (CSFRM) for equal variances.

TABLE 5. Confidence interval for wind direction data of Bayan Lepas

DATA	BAYAN LEPAS	
n	62	
Parameter	Estimate	Standard Error
$\hat{\alpha}_1$	1.3552	0.7474
$\hat{\alpha}_2$	1.4939	0.7212
$\hat{\beta}_1$	1.0811	0.5847
$\hat{\beta}_2$	1.4986	0.6040
$\hat{\omega}_1$	0.0017	0.1312
$\hat{\omega}_2$	0.0244	0.0987
$\hat{\kappa}$	11.8918	0.2441
95% CI for α_1	(0.0162, 3.1314)	
95% CI for α_2	(0.0065, 3.1308)	
95% CI for β_1	(1.0039, 1.1741)	
95% CI for β_2	(1.4003, 1.6428)	
95% CI for ω_1	(-0.0237, 0.0380)	
95% CI for ω_2	(-0.0034, 0.1419)	
95% CI for κ	(7.1167, 16.9481)	

CONCLUSION

This article proposed confidence interval for all parameters estimate of von Mises distribution in Circular Simultaneous Functional Relationship Model (CSFRM) for equal variances. The confidence interval is derived and verified by comparing it to both method of Normal Asymptotic Confidence Interval (NACI) and Bootstrap Confidence Interval (BCI). Its performance is assessed through simulation studies using different values of sample size and concentration parameter. Based on measuring performance coverage probability and expected length, the bootstrap method yields better results compared to when using normal asymptotic method. The practical value of the proposed confidence interval of bootstrap method have been shown when applied to real wind data set and the results showed it worked well with the model. It is also proven that the proposed confidence interval can be extended in cases where the error variance is not equal.

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