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The Solvability of the 3-D Elastic Wave Equations in Inhomogeneous Media

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ABSTRACT

In this research, the three-dimensional elastic wave equations with variable coefficients (i.e. propagate through inhomogeneous media) are solved with the application of the Fourier transform in the spatial coordinates. The wave equation is coupled variable coefficients PDEs whose solutions may have significant in engineering applications. The method utilizes the second order ODE as the baseline for obtaining the complete solution. The solution of second order ODEs is expressed in one integration because the variable coefficients are broken down into several functions and resulted in first order reduction. Moreover, the coupled equations are performed by the order reduction of the higher order ODEs into the second order. The extended procedure for integral equation is implemented for the solutions from the transformed wave equations to generate the explicit expression. It is shown that the proposed method of integral evaluation is resulted in finding the roots of polynomials. Hence, it is concluded that the solvability of the elastic wave equations is ensured by the proposed method.

Keywords: Wave equation; Inhomogeneous media; Reduction of order; Integral evaluation; Reduction of polynomial order

INTRODUCTION

The growing interests have been taking place in wave modeling fields. The transmittance and reflectance of sound wave with modulated speed is investigated by Mikhalevich & Streltsov (2009). It is shown that the generated parameters are determined significantly by wave intensities and phase shift. The high-speed acoustic wave with dissipation in saturated sediment is also considered (Naugolnykh & Esipov 2005). The consideration leads to the nonlinear evolution equation which the shock profile depends on the relaxation effects. The modified Kudraysov method with distinct integration schemes is utilized for ion sound wave (Seadawy et al. 2018). It is concluded that the approach is practically effective and can be exerted to several coupled PDEs. The similar method of reduction of the wave equation is found in Dzyuba & Romashko (2020). In this case, it is found that the speed of sound will have significant effect instead of pressure.

The geophysical geophysical problem of baroclinic wave packets is studied by Xie & Meng (2018). The one- and two-soliton solutions are obtained and the amplitude of the solitons become higher with the increasing parameters. An exact solution for the geophysical water wave is investigated (Henry 2013). It is noted that the vorticity is not affected for the constant underlying current. Meanwhile, the losses of coal measurement related to elastic wave is investigated. The low-frequency elastic wave with constant loss coefficient is considered and numerically solved to give the intrinsic absorption and scattering (Guo et al. 2020). The large number of databanks are produced by the strong motion instrument motion in earthquake and seismology (Stamatovska 2012). They may become the validating instrument for the theoretical and mathematical solutions of elastic wave equation because better results will be obtained if the instrument position network is permanent. The Sine-Gordon expansion method is implemented for generating the exact solutions of the coupled Drinfeld-Sokolov-Wilson equation (Tarbozan et al. 2018). The method is also assisted by the perturbation iteration algorithm and it is concluded that the method is powerful and reliable. The study of Boussinesq equation is also performed to produce solitary wave solutions. It is concluded that the method is useful for extracting the exact solutions for shallow water problem and other nonlinear evolution equations (Hossain et al. 2018). There are other related fields are studied which shows the importance of the wave equations and their solutions (Jleli et al. 2020; Wilk et al. 2017; Yu-Ting et al. 2013).

The considered problems depend on the medium and type of applications. The present study deals with the initial value problem of the three-dimensional elastic wave with variable coefficients. The considered problem is a system of linear PDE with possible spatially distributed forcing functions in time and *x, y, z* directions, which can be applied to the geophysical problems and seismology (Kanth 2008; Zhang et al. 1991; Pasternak & Dyskin 2008). Since the development of geological exploration for oil and gas including seismology need more description, indeed the study of the three-dimensional case is a challenging task. In fact, the knowledge of the time and spatial distributions by both analytical and numerical method has attracted researchers in many fields (Guidotti et al. 2006).

In this work, the governing equation is simplified by application of Fourier transform in the spatial directions. Starting from the x – displacement, the equation is solved analytically and is substituted sequentially in y – and z – displacements. Since the higher order ODE with variable coefficients is produced, the method of order reduction is developed in this research. After second order equation is achieved, the method for solving second order ODE with variable coefficients is investigated and proposed. The evaluation of integral is also presented to compute the obtained solutions.

PROBLEM FORMULATION

Consider the initial value problem of 3-dimensional wave equation with variable coefficients as in the following (Yang, 2014),

$$u_{tt}=\frac{\partial}{\partial x}\left[c_p^2\left(u_x+v_y+w_z\right)\right]-2\frac{\partial}{\partial x}\left[c_s^2\left(v_y+w_z\right)\right]+\frac{\partial}{\partial y}\left[c_s^2\left(u_y+v_x\right)\right]+\frac{\partial}{\partial z}\left[c_s^2\left(w_x+u_z\right)\right]+f_1 \tag{1a}$$

$$v_{tt}=\frac{\partial}{\partial y}\left[c_p^2\left(u_x+v_y+w_z\right)\right]-2\frac{\partial}{\partial y}\left[c_s^2\left(u_x+w_z\right)\right]+\frac{\partial}{\partial z}\left[c_s^2\left(v_z+w_y\right)\right]+\frac{\partial}{\partial x}\left[c_s^2\left(u_y+v_x\right)\right]+f_2 \tag{1b}$$

$$w_{tt}=\frac{\partial}{\partial z}\left[c_p^2\left(u_x+v_y+w_z\right)\right]-2\frac{\partial}{\partial z}\left[c_s^2\left(u_x+v_y\right)\right]+\frac{\partial}{\partial x}\left[c_s^2\left(w_x+u_z\right)\right]+\frac{\partial}{\partial y}\left[c_s^2\left(v_z+w_y\right)\right]+f_3 \tag{1c}$$

$$\begin{aligned} &u(x,y,z,0)=g_1(x,y,z), u_t(x,y,z,0)=g_2(x,y,z)\\ &v(x,y,z,0)=g_3(x,y,z), v_t(x,y,z,0)=g_4(x,y,z)\\ &w(x,y,z,0)=g_5(x,y,z), w_t(x,y,z,0)=g_6(x,y,z)\end{aligned} \tag{1d}$$

where $f_i\,(x, y, z, t)$ is the forcing function in each x, y and z directions, ρ is solid density, u, v and w are the displacements in x, y and z directions. $c_p=\sqrt{\frac{\lambda+2\mu}{\rho}}$ and $c_s=\sqrt{\frac{\mu}{\rho}}$ are the P-wave and μ shear wave velocities. The variant parameters λ and are elastic moduli of the solid which depend on the Young modulus and Poisson ratio. In this case the functions f_i and g_i are also smooth. In this case, equation (1) can be rewritten as,

$$\begin{aligned} u_{tt}&=c_p^2u_{xx}+c_s^2u_{yy}+c_s^2u_{zz}+\left(c_p^2-c_s^2\right)v_{xy}+\left(c_p^2-c_s^2\right)w_{xz}+c_{px}^2u_x+c_{sx}^2u_y+c_{sx}^2u_z+c_{sx}^2v_x+\left(c_{px}^2-2c_{sx}^2\right)v_y\\ &+c_{sx}^2w_x+c_{px}^2w_y-2c_{sx}^2w_z+f_1\end{aligned} \tag{2a}$$

$$\begin{aligned} v_{tt}&=c_s^2v_{xx}+c_p^2v_{yy}+c_s^2v_{zz}+\left(c_p^2-c_s^2\right)u_{xy}+\left(c_p^2-c_s^2\right)w_{yz}+\left(c_{py}^2-2c_{sy}^2\right)u_x+c_{sx}^2u_y+c_{sx}^2v_x+c_{py}^2v_y+c_{sz}^2v_z\\ &+c_{sz}^2w_y+\left(c_{py}^2-2c_{sy}^2\right)w_z+f_2\end{aligned} \tag{2b}$$

$$\begin{aligned} w_{tt}&=c_s^2w_{xx}+c_s^2w_{yy}+c_p^2w_{zz}+\left(c_p^2-c_s^2\right)u_{xz}+\left(c_p^2-c_s^2\right)v_{yz}+\left(c_{pz}^2-2c_{sz}^2\right)u_x+c_{sx}^2u_z+\left(c_{pz}^2-2c_{sz}^2\right)v_y+c_{sy}^2v_z\\ &+c_{sx}^2w_x+c_{sy}^2w_y+c_{pz}^2w_z+f_3\end{aligned} \tag{2c}$$

Considering the equation in x – direction, the Fourier transform, $H(k_i)=(2\pi)^{-\frac{3}{2}}\int_{x_i}h(x_i)e^{-ik_ix_i}dx_i$ of the spatial coordinate is given by,

$$\begin{aligned} &(2\pi)^{-\frac{3}{2}}U_{tt}=-c_p^2*k_1^2U-c_s^2*k_2^2U-c_s^2*k_3^2U\\ &-\left(c_p^2-c_s^2\right)*k_1k_2V-\left(c_p^2-c_s^2\right)*k_1k_3W-k_1c_p^2*k_1U\\ &-k_1c_s^2*k_2U-k_1c_s^2*k_3U-k_1c_s^2*k_1V-k_1c_p^2*k_2V\\ &+2k_1c_s^2*k_2V-k_1c_s^2*k_1W-k_1c_p^2*k_2W\\ &+2k_1c_s^2*k_3W+F_1\end{aligned} \tag{3a}$$

where the index i = j = 1, 2, 3 represents the spatial coordinates, (x, y, z). The convolution of the first term in r.h.s. of (3a), $c_p^2*k_1^2U=\int_l c_p^2(k_i-l_i)l_1^2U(l)dl$ can be rearranged as,

$$\int_{l_i}c_p^2\left(k_i-l_i\right)l_1^2U\left(l_i\right)dl_i=$$

$$\int_{l_j}\left(k_1-l_1\right)^2c_p^2\left(k_i-l_j\right)U\left(l_j\right)dl_j$$

which is only eligible when the range of $l_j = \frac{1}{2} l_i$. Applying the formulation to the rest of (3a) to produce,

$$\begin{aligned}(2\pi)^{-\frac{3}{2}} U_{tt} &= -k_1^2 c_p^2 * U - k_2^2 c_s^2 * U - k_3^2 c_s^2 * U \\ &- k_1 k_2 \left(c_p^2 - c_s^2\right) * V - k_1 k_3 \left(c_p^2 - c_s^2\right) * W \\ &- k_1^2 c_p^2 * U - k_1 k_2 c_s^2 * U - k_1 k_3 c_s^2 * U - k_1^2 c_s^2 *_1 V \\ &- k_1 k_2 c_p^2 * V + 2 k_1 k_2 c_s^2 * V - k_1^2 c_s^2 * W \\ &- k_1 k_2 c_p^2 * W + 2 k_1 k_3 c_s^2 * W + F_1\end{aligned} \tag{3b}$$

Performing the other transformation, $\tilde{I}(m_i) = (2\pi)^{-\frac{3}{2}} \int_{x_i} I(k_i) e^{-i m_i k_i} dk_i$ will produce the following equation,

$$\begin{aligned}(2\pi)^{-\frac{3}{2}} U_{tt} &= -a_1 * U - a_2 * U - a_3 * U - (a_4 - a_5) * V \\ &- (a_6 - a_7) * W - a_1 * U - a_5 * U - a_7 * U - a_8 * V \\ &- a_4 * V + 2a_5 * V - a_8 * W - a_4 * W + 2a_7 * W + F_1\end{aligned}$$

or

$$\begin{aligned}(2\pi)^{-3} U_{tt} &= -a_1 U - \widetilde{a_2} U - a_3 U - \left(\widetilde{a_4} - \widetilde{a_5}\right) V \\ &- \left(\widetilde{a_6} - \widetilde{a_7}\right) W - a_1 U - \widetilde{a_5} U - \widetilde{a_7} U - a_8 V - \widetilde{a_4} V \\ &+ 2\widetilde{a_5} V - a_8 W - \widetilde{a_4} W + 2\widetilde{a_7} W + F_1\end{aligned}$$

or

$$(2\pi)^{-3} U_{tt} = -b_1 U + b_2 V + b_3 W + F_1 \tag{4a}$$

The same procedure is applied to the y and z – directions and produce,

$$(2\pi)^{-3} V_{tt} = -2b_4 V + 2b_5 U + 2b_6 W + \widetilde{F_2} \tag{4b}$$

$$(2\pi)^{-3} \widetilde{W_{tt}} = -2b_7 W + 2b_8 U + 2b_9 V + \widetilde{F_3} \tag{4c}$$

where,

$$\begin{aligned}&a_1 = k_1^2 c_p^2, a_2 = k_2^2 c_s^2, a_3 = k_3^2 c_s^2, a_4 = k_1 k_2 c_p^2, \\ &a_5 = k_1 k_2 c_s^2, a_6 = k_1 k_3 c_p^2, a_7 = k_1 k_3 c_s^2, \\ &a_8 = k_1^2 c_s^2, a_9 = k_2^2 c_p^2, a_{10} = k_2 k_3 c_p^2, \\ &a_{11} = k_2 k_3 c_s^2, a_{12} = k_3^2 c_p^2\end{aligned}$$

and

$$\begin{aligned}&b_1 = 2a_1 + \widetilde{a_2} + a_3 + \widetilde{a_5} + \widetilde{a_7}, b_2 = 3\widetilde{a_5} - 2\widetilde{a_4} - a_8, \\ &b_3 = 3\widetilde{a_7} - \widetilde{a_4} - \widetilde{a_6} - a_8, b_4 = a_8 + \widetilde{a_9} + a_3, \\ &b_5 = \widetilde{a_5} - \widetilde{a_4}, b_6 = \widetilde{a_{11}} - \widetilde{a_{10}}, b_7 = a_8 + \widetilde{a_2} + \widetilde{a_{12}}, \\ &b_8 = \widetilde{a_7} - \widetilde{a_6}, b_9 = \widetilde{a_{11}} - \widetilde{a_{10}}\end{aligned}$$

THE SOLUTION OF THE VARIABLE COEFFICIENTS ELASTIC WAVE EQUATIONS

Consider the homogenous term of equation (4a), we have to solve the second order ODE with variable coefficients as a first step.

SOLUTION OF SECOND ORDER ODE WITH VARIABLE COEFFICIENTS

The ODE is defined by,

$$y_{tt} + a_1 y_t + a_2 y = 0 \tag{5a}$$

Note that the reader should not be confused with the same symbols as they represent different functions. Let $a_1 = b_1 + \frac{b_{2t}}{b_2}$ and $a_2 = b_3 + b_1 \frac{b_{4t}}{b_4}$, to produce,

$$\frac{1}{b_2}\left(b_2 y_t\right)_t + \frac{b_1}{b_4}\left(b_4 y\right)_t + b_3 y = 0 \text{ or}$$

$$\frac{1}{b_2}\left[\frac{b_2}{b_4} z_t + b_2 \left(\frac{1}{b_4}\right)_t z\right]_t + \frac{b_1}{b_4} z_t + \frac{b_3}{b_4} z = 0 \tag{5b}$$

where, $z = b_4 y$. Multiply by a function β, to produce

$$\frac{\beta}{b_2}\left[\frac{b_2}{b_4} z_t + b_2 \left(\frac{1}{b_4}\right)_t z\right]_t + \beta \frac{b_1}{b_4} z_t + \beta \frac{b_3}{b_4} z = 0$$

Suppose that, $\beta_t \frac{b_1}{b_4} = \beta \frac{b_3}{b_4}$ then $\beta = C_1 e^{\int_t \frac{b_3}{b_1} dt}$, the above equation becomes,

$$\frac{\beta}{b_2}\left[\frac{b_2}{b_4} z_t + b_2 \left(\frac{1}{b_4}\right)_t z\right]_t + \frac{b_1}{b_4}\left(\beta z\right)_t = 0 \quad \text{or}$$

$$\frac{\beta}{b_2}\left\{\frac{b_2}{\beta b_4} A_t + \left[\frac{b_2}{b_4}\left(\frac{1}{\beta}\right)_t + \frac{b_2}{\beta}\left(\frac{1}{b_4}\right)_t\right] A\right\}_t + \frac{b_1}{b_4} A_t = 0 \tag{6a}$$

with, $A = \beta z = \beta b_4 y$. It is assumed that the following relation is hold, $\left[\frac{b_2}{b_4}\left(\frac{1}{\beta}\right)_t + \frac{b_2}{\beta}\left(\frac{1}{b_4}\right)_t\right]_t = 0$, thus the above equation becomes,

$$\begin{aligned}&-\left(\frac{b_3}{b_1}\right)_t + \left(\frac{b_3}{b_1}\right)^2 - 2\frac{b_{2t}}{b_2}\frac{b_3}{b_1} - 2b_4 \frac{b_3}{b_1}\left(\frac{1}{b_4}\right)_t \\ &+ b_4 \left(\frac{1}{b_4}\right)_{tt} = 0\end{aligned} \tag{6b}$$

Then, we have the following expression,

$$\frac{b_{2t}}{b_2} = \frac{-\left(\frac{b_3}{b_1}\right)_t + \left(\frac{b_3}{b_1}\right)^2 - 2b_4 \frac{b_3}{b_1}\left(\frac{1}{b_4}\right)_t + b_4 \left(\frac{1}{b_4}\right)_{tt}}{2\frac{b_3}{b_1}} \tag{6c}$$

Recall that, $a_1 = b_1 + \frac{b_{2t}}{b_2}$, substituting (6c),

$$2a_1\frac{b_3}{b_1} = 2b_3 - \left(\frac{b_3}{b_1}\right)_t + \left(\frac{b_3}{b_1}\right)^2 - 2b_4\frac{b_3}{b_1}\left(\frac{1}{b_4}\right)_t + b_4\left(\frac{1}{b_4}\right)_{tt}$$

Replacing b_3 with the functions taken from, $a_2 = b_3 + b_1\frac{b_{4t}}{b_4}$

$$\left(\frac{a_2}{b_1}-\frac{b_{4t}}{b_4}\right)_t - \left(\frac{a_2}{b_1}-\frac{b_{4t}}{b_4}\right)^2 + 2a_1\left(\frac{a_2}{b_1}-\frac{b_{4t}}{b_4}\right) =$$
$$2b_1\left(\frac{a_2}{b_1}-\frac{b_{4t}}{b_4}\right) - 2b_4\left(\frac{a_2}{b_1}-\frac{b_{4t}}{b_4}\right)\left(\frac{1}{b_4}\right)_t \tag{4d}$$
$$+b_4\left(\frac{1}{b_4}\right)_{tt} = 0$$

Let, $\frac{a_2}{b_1}-\frac{b_{4t}}{b_4} = 0$, then $b_4 = C_2 e^{\int_t \frac{a_2}{b_1}dt}$, the remaining equation is,

$$b_4\left(\frac{1}{b_4}\right)_{tt} = 0 \quad \text{or} \quad \frac{a_2}{b_1}e^{-\int_t \frac{a_2}{b_1}dt} = C \quad \text{or}$$
$$\left(\frac{a_2}{b_1}\right)_t - \left(\frac{a_2}{b_1}\right)^2 = 0 \tag{7a}$$

The solution for b_1 is then,

$$b_1 = -a_2\left(C_3 t + C_4\right) \tag{7b}$$

It is also clear that, $\beta = C_1$ and $b_2 = C_5 e^{\int_t a_1 + a_2(C_3 t + C_4)dt}$. The ODE is thus become,

$$\frac{1}{b_4}A_{tt} + \left(\frac{b_{2t}}{b_2 b_4} - \frac{\beta_t}{\beta b_4} - \frac{b_{4t}}{b_4^2} + C\frac{\beta}{b_2} + \frac{b_1}{b_4}\right)A_t \text{ or}$$

$$A_{tt} + \left(a_1 - \frac{b_{4t}}{b_4} + \frac{C_6}{b_2}b_4\right)A_t = 0$$

The solution for A and y are,

$$A = C_7\left\{\int_t \frac{1}{\left(C_3 t + C_4\right)}\exp - \int_t\left[a_1 + \frac{C_6}{\left(C_3 t + C_4\right)}e^{-\int_t\left[a_1 + a_2\left(C_3 t + C_4\right)\right]dt}\right]dtdt + C_8\right\} \text{ and}$$
$$y = \left(C_3 t + C_4\right)\left\{\int_t \frac{1}{\left(C_3 t + C_4\right)}\exp - \int_t\left[a_1 + \frac{C_6}{\left(C_3 t + C_4\right)}e^{-\int_t\left[a_1 + a_2\left(C_3 t + C_4\right)\right]dt}\right]dtdt + C_8\right\} \tag{7c}$$

Thus, the homogeneous solution of (4a) is,

$$\widetilde{U_h} = \left(X_3 t + X_4\right)\left\{\int_t \frac{1}{\left(X_3 t + X_4\right)}\exp - \int_t\left[\frac{X_6}{\left(X_3 t + X_4\right)}e^{(2\pi)^3\int_t b_1\left(X_3 t + X_4\right)dt}\right]dtdt + X_8\right\} \tag{8a}$$

Note that $X_i = X_i(m_1, m_2, m_3)$ as the integration constants w.r.t. time and may generate the complex waves propagation. The nonhomogeneous solution is determined by,

$$\widetilde{U_p} = (2\pi)^3\int_t \frac{1}{\widetilde{U_h^2}}\left[\int_t \widetilde{U_h}\left(b_2 V + b_3 W + F_1\right)dt\right]dt + X_3$$

And the complete solution is,

$$U = \widetilde{U_h}\widetilde{U_p} =$$
$$(2\pi)^3\widetilde{U_h}\int_t \frac{1}{\widetilde{U_h^2}}\left[\int_t \widetilde{U_h}\left(b_2 V + b_3 W + F_1\right)dt\right]dt \tag{8b}$$
$$+X_3\widetilde{U_h}$$

The next step is to substitute the results into (4b) as follows,

$$(2\pi)^{-3}V_{tt} = -2b_4 V$$
$$+2(2\pi)^3 b_5\widetilde{U_h}\int_t \frac{1}{\widetilde{U_h^2}}\left[\int_t \widetilde{U_h}\left(b_2 V + b_3 W + F_1\right)dt\right]dt \tag{8c}$$
$$+2b_5 X_3\widetilde{U_h} + 2b_6 W + \widetilde{F_2}$$

Differentiate twice and rearranging the coefficients

$$(2\pi)^{-3}\frac{1}{2b_5\widetilde{U_h}}V_{ttt}+(2\pi)^{-3}\left(\frac{1}{2b_5\widetilde{U_h}}\right)_t V_{tt}=-\frac{b_4}{b_5\widetilde{U_h}}V_t-\left(\frac{b_4}{b_5\widetilde{U_h}}\right)_t V+\frac{(2\pi)^{-}}{\widetilde{U_h^2}}\left[\int_t\widetilde{U_h}\left(b_2V+b_3W+F_1\right)dt\right]$$

$$+\frac{b_6}{b_5\widetilde{U_h}}W_t+\left(\frac{b_6}{b_5\widetilde{U_h}}\right)_t W+\left(\frac{\widetilde{F_2}}{2b_5\widetilde{U_h}}\right)_t$$

or

$$(2\pi)^{-3}\frac{\widetilde{U_h}}{2b_5}V_{tttt}+(2\pi)^{-3}\left[\left(\frac{\widetilde{U_h}}{2b_5}\right)_t+\widetilde{U_h^2}\left(\frac{1}{2b_5\widetilde{U_h}}\right)_t\right]V_{ttt}+\left\{(2\pi)^{-3}\left[\widetilde{U_h^2}\left(\frac{1}{2b_5\widetilde{U_h}}\right)_t\right]_t+\frac{b_4\widetilde{U_h}}{b_5}\right\}V_{tt}=$$

$$-\left[\left(\frac{b_4\widetilde{U_h}}{b_5}\right)_t+\widetilde{U_h^2}\left(\frac{b_4}{b_5\widetilde{U_h}}\right)_t\right]V_t+\left\{(2\pi)^3 b_2\widetilde{U_h}-\left[\widetilde{U_h^2}\left(\frac{b_4}{b_5\widetilde{U_h}}\right)_t\right]_t\right\}V+\widetilde{U_h^2}\left(\frac{\widetilde{F_2}}{2b_5\widetilde{U_h}}\right)_t+(2\pi)^3\widetilde{U_h}F_1 \quad (9a)$$

$$+\frac{b_6\widetilde{U_h}}{b_5}W_{tt}+\left[\left(\frac{b_6\widetilde{U_h}}{b_5}\right)_t+\widetilde{U_h^2}\left(\frac{b_6}{b_5\widetilde{U_h}}\right)_t\right]W_t+\left\{(2\pi)^3 b_3\widetilde{U_h}+\left[\widetilde{U_h^2}\left(\frac{b_6}{b_5\widetilde{U_h}}\right)_t\right]_t\right\}W$$

Equation (9a) is fourth order ODE by twice differentiation and written in implicit form as,

$$V_{tttt}+\alpha_1V_{ttt}+\alpha_2V_{tt}+\alpha_3V_t+\alpha_4V=\alpha_5 \quad (9b)$$

with,

$$\alpha_5=(2\pi)^3\left[\left(\frac{b_6\widetilde{U_h}}{b_5}\right)_t+\widetilde{U_h^2}\left(\frac{b_6}{b_5\widetilde{U_h}}\right)_t\right]W_t$$

$$+(2\pi)^3\left\{(2\pi)^3b_3\widetilde{U_h}+\left[\widetilde{U_h^2}\left(\frac{b_6}{b_5\widetilde{U_h}}\right)_t\right]_t\right\}W$$

$$+(2\pi)^3\widetilde{U_h^2}\left(\frac{\widetilde{F_2}}{2b_5\widetilde{U_h}}\right)_t+(2\pi)^6\widetilde{U_h}F_1$$

$$+(2\pi)^3\frac{b_6\widetilde{U_h}}{b_5}W_{tt}$$

REDUCTION OF ORDER

In this section the variable coefficient fourth order ODE is reduced into the third order and then second order ODE. Now consider the fourth order ODE,

$$y_{xxxx}+a_1y_{xxx}+a_2y_{xx}+a_3y_x+a_4y=\varphi$$

Introducing the coefficient decomposition as,

$$a_1=b_1+\frac{a_{5x}}{a_5},\ a_2=b_2+b_1\frac{a_{6x}}{a_6}\ \text{and}$$

$$a_3=b_3+b_2\frac{a_{7x}}{a_7} \quad (10a)$$

Note that symbols a_i and b_i are different from previous section and the reader should not be confused. The fourth order ODE becomes,

$$\frac{1}{a_5}\left(a_5y_{xxx}\right)_x+\frac{b_1}{a_6}\left(a_6y_{xx}\right)_x+\frac{b_2}{a_7}\left(a_7y_x\right)_x+b_3y_x+a_4y=\varphi$$

Multiplying by an arbitrary function γ_1 to give,

$$\frac{\gamma_1}{a_5}\left(a_5y_{xxx}\right)_x+\frac{\gamma_1b_1}{a_6}\left(a_6y_{xx}\right)_x+\frac{\gamma_1b_2}{a_7}\left(a_7y_x\right)_x+\gamma_1b_3y_x+\gamma_1a_4y=\gamma_1\varphi \quad (10b)$$

Take the following relation,

$$\gamma_{1x}b_3=\gamma_1a_4 \ \text{then,}\ \gamma_1=e^{\int_x\frac{a_4}{b_3}dx} \quad (10c)$$

Equation (10c) is transformed as,

$$\frac{\gamma_1}{a_5}\left(a_5y_{xxx}\right)_x+\frac{\gamma_1b_1}{a_6}\left(a_6y_{xx}\right)_x+\frac{\gamma_1b_2}{a_7}\left(a_7y_x\right)_x+b_3\left(e^{\int_x\frac{a_4}{b_3}dx}y\right)_x=\gamma_1\varphi$$

Let us assume that,

$$e^{\int_x\frac{a_4}{b_3}dx}y=h_1,\ \text{and}\ y=h_1e^{-\int_x\frac{a_4}{b_3}dx} \quad (10d)$$

Expanding equation (10b) as,

$$\frac{\gamma_1}{a_5}\left\{a_5 e^{-\int_x \frac{a_4}{b_3}dx}\left[h_{1xxx}+3h_{1xx}\left(-\frac{a_4}{b_3}\right)+3h_{1x}\left[\left(-\frac{a_4}{b_3}\right)_x+\left(\frac{a_4}{b_3}\right)^2\right]+h_1\left[\left(-\frac{a_4}{b_3}\right)_{xx}+3\left(\frac{a_4}{b_3}\right)_x\left(\frac{a_4}{b_3}\right)-\left(\frac{a_4}{b_3}\right)^3\right]\right]\right\}_x+$$

$$\frac{\gamma_1 b_1}{a_6}\left\{a_6 e^{-\int_x \frac{a_4}{b_3}dx}\left[h_{1xx}+2h_{1x}\left(-\frac{a_4}{b_3}\right)+h_1\left[\left(-\frac{a_4}{b_3}\right)_x+\left(\frac{a_4}{b_3}\right)^2\right]\right]\right\}_x+\frac{\gamma_1 b_2}{a_7}\left\{a_7 e^{-\int_x \frac{a_4}{b_3}dx}\left[h_{1x}+h_1\left(-\frac{a_4}{b_3}\right)\right]\right\}_x+b_3h_x=\gamma_1\varphi$$

Performing the following relation,

$$\frac{\gamma_1}{a_5}\left\{a_5\left[\left(-\frac{a_4}{b_3}\right)_{xx}+3\left(\frac{a_4}{b_3}\right)_x\left(\frac{a_4}{b_3}\right)-\left(\frac{a_4}{b_3}\right)^3\right]e^{-\int_x \frac{a_4}{b_3}dx}\right\}_x+\frac{\gamma_1 b_1}{a_6}\left\{a_6\left[\left(-\frac{a_4}{b_3}\right)_x+\left(\frac{a_4}{b_3}\right)^2\right]e^{-\int_x \frac{a_4}{b_3}dx}\right\}_x$$

$$+\frac{\gamma_1 b_2}{a_7}\left[a_7\left(-\frac{a_4}{b_3}\right)e^{-\int_x \frac{a_4}{b_3}dx}\right]_x=0$$

or

$$\frac{a_{5x}}{a_5}\left[\left(-\frac{a_4}{b_3}\right)_{xx}+3\left(\frac{a_4}{b_3}\right)_x\left(\frac{a_4}{b_3}\right)-\left(\frac{a_4}{b_3}\right)^3\right]-\left(\frac{a_4}{b_3}\right)_{xxx}+4\left(\frac{a_4}{b_3}\right)_{xx}\left(\frac{a_4}{b_3}\right)+3\left(\frac{a_4}{b_3}\right)_x^2-5\left(\frac{a_4}{b_3}\right)_x\left(\frac{a_4}{b_3}\right)^2+\left(\frac{a_4}{b_3}\right)^4$$

$$\frac{a_{6x}b_1}{a_6}\left[\left(-\frac{a_4}{b_3}\right)_x+\left(\frac{a_4}{b_3}\right)^2\right]+b_1\left[\left(-\frac{a_4}{b_3}\right)_{xx}+3\left(\frac{a_4}{b_3}\right)_x\left(\frac{a_4}{b_3}\right)-\left(\frac{a_4}{b_3}\right)^3\right]-\frac{a_{7x}b_2}{a_7}\left(\frac{a_4}{b_3}\right)+b_2\left[\left(-\frac{a_4}{b_3}\right)_x+\left(\frac{a_4}{b_3}\right)^2\right]=0 \quad \text{(11a)}$$

Suppose that b_3 is given, then $\frac{a_4}{b_3}$ known, thus b_2 can be determined form (11a) as,

$$b_2=\frac{f_1+b_1f_3+b_1\frac{a_{6x}}{a_6}f_2+\frac{a_{5x}}{a_5}f_3}{\frac{a_{7x}}{a_7}f_4-f_2}$$

Substituting (11b) into the third relation of (10a) to give b_1 as in the following,

$$(a_3-b_3)\left(\frac{a_{7x}}{a_7}f_4-f_2\right)=f_1\frac{a_{7x}}{a_7}+\frac{a_{5x}}{a_5}\frac{a_{7x}}{a_7}f_3$$

or

$$+b_1\left(\frac{a_{7x}}{a_7}f_3+\frac{a_{6x}}{a_6}\frac{a_{7x}}{a_7}f_2\right)$$

$$b_1=\frac{(a_3-b_3)\left(\frac{a_{7x}}{a_7}f_4-f_2\right)-f_1\frac{a_{7x}}{a_7}-\frac{a_{5x}}{a_5}\frac{a_{7x}}{a_7}f_3}{\frac{a_{7x}}{a_7}f_3+\frac{a_{6x}}{a_6}\frac{a_{7x}}{a_7}f_2}$$

Continuing into the second relation of (10a),

$$\left(a_2\frac{a_{7x}}{a_7}f_4-\frac{a_{5x}}{a_5}f_3-a_2f_2-f_1\right)\left(\frac{a_{7x}}{a_7}f_3+\frac{a_{6x}}{a_6}\frac{a_{7x}}{a_7}f_2\right)=(a_3-b_3)\left(\frac{a_{7x}}{a_7}f_4-f_2\right)-f_1\frac{a_{7x}}{a_7}-\frac{a_{5x}}{a_5}\frac{a_{7x}}{a_7}f_3\left(f_3+\frac{a_{6x}}{a_6}\frac{a_{7x}}{a_7}f_4\right)$$

or

$$\frac{a_{5x}}{a_5}=\frac{\left(a_2f_2+f_1-a_2\frac{a_{7x}}{a_7}f_4\right)\left(\frac{a_{7x}}{a_7}f_3+\frac{a_{6x}}{a_6}\frac{a_{7x}}{a_7}f_2\right)+(a_3-b_3)\left(\frac{a_{7x}}{a_7}f_4-f_2\right)-f_1\frac{a_{7x}}{a_7}}{\frac{a_{6x}}{a_6}\left(\frac{a_{7x}}{a_7}\right)^2f_3f_4-\frac{a_{6x}}{a_6}\frac{a_{7x}}{a_7}f_2f_3}$$

Thus, the first relation will determine a_6 or a_7 as,

$$\left[a_1\frac{a_{7x}}{a_7}f_3+a_1\frac{a_{6x}}{a_6}\frac{a_{7x}}{a_7}f_2-(a_3-b_3)\left(\frac{a_{7x}}{a_7}f_4-f_2\right)+f_1\frac{a_{7x}}{a_7}\right]\left[\frac{a_{6x}}{a_6}\left(\frac{a_{7x}}{a_7}\right)^2 f_3f_4-\frac{a_{6x}}{a_6}\frac{a_{7x}}{a_7}f_2f_3\right]=$$
$$+\left[\left(a_2f_2+f_1-a_2\frac{a_{7x}}{a_7}f_4\right)\left(\frac{a_{7x}}{a_7}f_3+\frac{a_{6x}}{a_6}\frac{a_{7x}}{a_7}f_2\right)+(a_3-b_3)\left(\frac{a_{7x}}{a_7}f_4-f_2\right)-f_1\frac{a_{7x}}{a_7}\right]\left(\frac{a_{6x}}{a_6}\frac{a_{7x}}{a_7}f_2\right)$$

$$\frac{a_{6x}}{a_6}\left[a_1\left(\frac{a_{7x}}{a_7}\right)^2 f_2f_3f_4+a_2\left(\frac{a_{7x}}{a_7}\right)^2 f_2^2f_4-a_1\frac{a_{7x}}{a_7}f_2^2f_3-a_2\frac{a_{7x}}{a_7}f_2^3-\frac{a_{7x}}{a_7}f_1f_2^2\right]=$$

or $$\left[(a_3-b_3)\left(\frac{a_{7x}}{a_7}f_4-f_2\right)-a_1\frac{a_{7x}}{a_7}f_3-f_1\frac{a_{7x}}{a_7}\right]\left(\frac{a_{7x}}{a_7}f_3f_4-f_2f_3\right)+(a_3-b_3)\left(\frac{a_{7x}}{a_7}f_2f_4-f_2^2\right)$$
$$+a_2\frac{a_{7x}}{a_7}f_2^2f_3+\frac{a_{7x}}{a_7}f_1f_2f_3-a_2\left(\frac{a_{7x}}{a_7}\right)^2 f_2f_3f_4-\frac{a_{7x}}{a_7}f_1f_2$$

Therefore, the fourth order equation is reduced into,

$$j_{1xxx}+a_8j_{1xx}+a_9j_{1x}+a_{10}j_1=\gamma_1\varphi \qquad (11c)$$

where, $j_1 = h_1x$.

Repeat the procedure for third order ODE as in the following,

$$j_{1xxx}+a_8j_{1xx}+a_9j_{1x}+a_{10}j_1=\gamma_1\varphi$$

Suppose that the following relations are hold,

$$a_8=b_4+\frac{a_{11x}}{a_{11}} \text{ and } a_9=b_5+b_4\frac{a_{12x}}{a_{12}} \qquad (12a)$$

Thus, the following relation is obtained,

$$\frac{1}{a_{11}}\left(a_{11}j_{1xx}\right)_x+\frac{b_4}{a_{12}}\left(a_{12}j_{1x}\right)_x+b_5j_{1x}+a_{10}j_1=\gamma_1\varphi$$

Multiply by an arbitrary function γ_2 to generate,

$$\frac{\gamma_2}{a_{11}}\left(a_{12}j_{1xx}\right)_x+\frac{\gamma_2b_4}{a_{12}}\left(a_{12}j_{1x}\right)_x+\gamma_2b_5j_{1x}$$
$$+\gamma_2a_{10}j_1=\gamma_2\gamma_1\varphi \qquad (12b)$$

Suppose that the following expression is satisfied,

$$\gamma_{2x}b_5=\gamma_2a_{10} \text{ then, } \gamma_2=e^{\int_x\frac{a_{10}}{b_5}dx} \qquad (12c)$$

Equation (12b) is rewritten as,

$$\frac{\gamma_2}{a_{11}}\left(a_{11}j_{1xx}\right)_x+\frac{\gamma_2b_4}{a_{12}}\left(a_{12}j_{1x}\right)_x$$
$$+b_5\left(e^{\int_x\frac{a_{10}}{b_5}dx}j_1\right)_x=\gamma_2\gamma_1\varphi$$

Suppose that,

$$e^{\int_x\frac{a_3}{b_2}dx}j_1=h_2, \text{ and } j_1=h_2e^{-\int_x\frac{a_{10}}{b_5}dx} \qquad (13a)$$

Therefore, equation (12b) can be expanded as,

$$\frac{\gamma_2}{a_{11}}\left\{a_{11}e^{-\int_x\frac{a_{10}}{b_5}dx}\left[\begin{matrix}h_{2xx}+2h_{2x}\left(-\frac{a_{10}}{b_5}\right)\\ +h_2\left[\left(-\frac{a_{10}}{b_5}\right)_x+\left(\frac{a_{10}}{b_5}\right)^2\right]\end{matrix}\right]\right\}_x$$
$$+\frac{\gamma_2b_4}{a_{12}}\left\{a_{12}e^{-\int_x\frac{a_{10}}{b_5}dx}\left[h_{2x}+h_2\left(-\frac{a_{10}}{b_5}\right)\right]\right\}_x$$
$$+b_5h_{2x}=\gamma_2\gamma_1\varphi$$

Differentiate the above equation once again and relate,

$$\frac{\gamma_2}{a_{11}}\left\{a_{11}\left[\left(-\frac{a_{10}}{b_5}\right)_x+\left(\frac{a_{10}}{b_5}\right)^2\right]e^{-\int_x\frac{a_{10}}{b_5}dx}\right\}_x \text{ or}$$
$$+\frac{\gamma_2b_4}{a_{12}}\left[a_{12}\left(-\frac{a_{10}}{b_5}\right)e^{-\int_x\frac{a_{10}}{b_5}dx}\right]_x=0$$

$$\frac{a_{11x}}{a_{11}}\left[\left(-\frac{a_{10}}{b_5}\right)_x+\left(\frac{a_{10}}{b_5}\right)^2\right]+\left(-\frac{a_{10}}{b_5}\right)_{xx}$$
$$+3\left(\frac{a_{10}}{b_5}\right)_x\left(\frac{a_{10}}{b_5}\right)^2+\left(\frac{a_{10}}{b_5}\right)^3-b_4\frac{a_{12x}}{a_{12}}\left(\frac{a_{10}}{b_5}\right) \qquad (13b)$$
$$+b_4\left[\left(-\frac{a_{10}}{b_5}\right)_x+\left(\frac{a_{10}}{b_5}\right)^2\right]=0$$

Now assume that b_5 is given, thus $\frac{a_{10}}{b_5}$ is determined, and the relation, b_4 can be determined from (13b) as,

$$b_4 = \frac{f_5 + \frac{a_{11x}}{a_{11}} f_6}{\frac{a_{12x}}{a_{12}} f_7 - f_6} \tag{13c}$$

Substituting into the second equation of (12a) to give the expression of b_4 as a function of $\frac{a_{11x}}{a_{11}}$.

$$\left(a_9 - b_5\right) \frac{a_{12x}}{a_{12}} f_7 - \left(a_9 - b_5\right) f_6 = \frac{a_{12x}}{a_{12}} f_5 + \frac{a_{11x}}{a_{11}} \frac{a_{12x}}{a_{12}} f_6 \quad \text{or}$$

$$\frac{a_{11x}}{a_{11}} = \frac{\left(a_9 - b_5\right) \frac{a_{12x}}{a_{12}} f_7 - \frac{a_{12x}}{a_{12}} f_5 - \left(a_9 - b_5\right) f_6}{\frac{a_{12x}}{a_{12}} f_6} \tag{13d}$$

Performing the resulting expression into the first equation of (12a) to generate a_{12}.

$$a_8 \left(\frac{a_{12x}}{a_{12}}\right)^2 f_6 f_7 - a_8 \frac{a_{12x}}{a_{12}} f_6^2 = \frac{a_{12x}}{a_{12}} f_5 f_6 + \left(a_9 - b_5\right)\left(\frac{a_{12x}}{a_{12}}\right)^2 f_7^2 - \left(\frac{a_{12x}}{a_{12}}\right)^2 f_5 f_7 - \left(a_9 - b_5\right) \frac{a_{12x}}{a_{12}} f_6 f_7 \quad \text{or}$$

$$\frac{a_{12x}}{a_{12}} = \frac{a_8 f_6^2 + f_5 f_6 - \left(a_9 - b_5\right) f_6 f_7}{a_8 f_6 f_7 + f_5 f_7 - \left(a_9 - b_5\right) f_7^2} \tag{13e}$$

Therefore, equation (11c) is reduced into,

$$h_{2xxx} + a_{13} h_{2xx} + a_{14} h_{2x} = \gamma_2 \gamma_1 \varphi \tag{14a}$$

and solvable by transforming into the Riccati equation as in the previous section. In this case, the solution of (9b) that reduced into the same form of (5a) is,

$$h_2 = \left(2\pi\right)^3 \int_t \left[j_{2h} \int_t \frac{1}{j_{2h}^2} e^{-\int_t \alpha_{13} dt} \left(\int_t j_{2h} \gamma_2 \gamma_1 \alpha_5 e^{\int_t \alpha_{13} dt} dt \right) dt \right] dt + \int_t X_4 j_{2h} dt + X_5 \quad \text{or}$$

$$\begin{aligned} V &= \left(2\pi\right)^3 e^{-\int_t \frac{\alpha_4}{b_3} dt} \int_t \left\{ e^{-\int_t \frac{\alpha_{10}}{b_5} dt} \int_t \left[j_{2h} \int_t \frac{1}{j_{2h}^2} e^{-\int_t \alpha_{13} dt} \left(\int_t j_{2h} \gamma_2 \gamma_1 \alpha_5 e^{\int_t \alpha_{13} dt} dt \right) dt \right] dt \right\} dt \\ &+ e^{-\int_t \frac{\alpha_4}{b_3} dt} \int_t \left[e^{-\int_t \frac{\alpha_{10}}{b_5} dt} \left(\int_t X_4 j_{2h} dt \right) \right] dt + e^{-\int_t \frac{\alpha_4}{b_3} dt} \int_t X_5 e^{-\int_t \frac{\alpha_{10}}{b_5} dt} dt + X_6 e^{-\int_t \frac{\alpha_4}{b_3} dt} \end{aligned} \tag{14b}$$

The final step is substituting equations (8c) into (4c) as in the following,

$$\left(2\pi\right)^{-3} \widetilde{W_{tt}} = -2b_7 W + 2b_8 X_3 \widetilde{U_h} + 2b_9 V + \widetilde{F_3} + 2\left(2\pi\right)^3 b_8 \widetilde{U_h} \int_t \frac{1}{\widetilde{U_h^2}} \left[\int_t \widetilde{U_h} \left(b_2 V + b_3 W + F_1\right) dt \right] dt$$

Rearranging the coefficients and differentiating the above equation,

$$\begin{aligned} &\left(2\pi\right)^{-3} \frac{1}{2b_8} \widetilde{U_h} W_{tttt} + \left(2\pi\right)^{-3} \left[\left(\frac{1}{2b_8} \widetilde{U_h} \right)_t + \left(\frac{1}{2b_8 \widetilde{U_h}} \right)_t \widetilde{U_h^2} \right] W_{ttt} + \left\{ \left(2\pi\right)^{-3} \left[\left(\frac{1}{2b_8 \widetilde{U_h}} \right)_t \widetilde{U_h^2} \right]_t + \frac{b_7}{b_8} \widetilde{U_h} \right\} \widetilde{W_{tt}} \\ &+ \left[\left(\frac{b_7}{b_8} \widetilde{U_h} \right)_t + \widetilde{U_h^2} \left(\frac{b_7}{b_8 \widetilde{U_h}} \right)_t \right] \widetilde{W_t} + \left\{ \left[\widetilde{U_h^2} \left(\frac{b_7}{b_8 \widetilde{U_h}} \right)_t \right]_t - \left(2\pi\right)^3 b_3 \widetilde{U_h} \right\} W = \frac{b_9}{b_8} \widetilde{U_h} \widetilde{V_{tt}} + \left[\left(\frac{b_9}{b_8} \widetilde{U_h} \right)_t + \left(\frac{b_9}{b_8 \widetilde{U_h}} \right)_t \widetilde{U_h^2} \right] V_t \\ &+ \left\{ \left[\left(\frac{b_9}{b_8 \widetilde{U_h}} \right)_t \widetilde{U_h^2} \right]_t + \left(2\pi\right)^3 b_2 \widetilde{U_h} \right\} V + \left[\left(\frac{1}{2b_8 \widetilde{U_h}} \widetilde{F_3} \right)_t \widetilde{U_h^2} \right]_t + \left(2\pi\right)^3 \widetilde{U_h} F_1 \end{aligned} \tag{14c}$$

Performing (14b) into (14c), differentiating and rearranging the coefficients sequentially will produce the equation for W as a tenth order ODE. Performing the reduction of the order by the method illustrated in (10 – 13), the solution is also obtained for W.

REMARKS ON INTEGRAL EVALUATION

The obtained solutions of U, V and W have to be transformed back into the spatial domain by inverse Fourier transform. In this section, the method for integral evaluation is proposed for producing the explicit expression on al solutions. Consider the integral G as follows,

$$G=\int_x Ddx \tag{15a}$$

The integral will be evaluated as,

$$\int_x Ddx=\int_x D\frac{1}{V_x}dV=\int_x \ln VdV \text{ or} \tag{15b}$$

$$D=V_x\ln V$$

Assume that, $V=e^u$, then

$$D=u_xue^u \text{ or } \frac{D_x}{D}=\frac{u_{xx}}{u_x}+\frac{u_x}{u}+u_x \text{ or}$$

$$\left(\frac{1}{u_x}\right)_x=-\frac{D_x}{D}\frac{1}{u_x}+1+\frac{1}{u}$$

The solution is,

$$\frac{1}{u_x}=\frac{1}{D}\left(\int_x\frac{D}{u}+Ddx+C\right) \tag{15c}$$

Let, $u=-\frac{1}{v}$, then

$$v^2=-\frac{1}{D}v_x\left(\int_x Dv+Ddx+C\right) \tag{15d}$$

Let, $w=\int_x Dvdx$, thus the above equation become,

$$w_x^2+ww_{xx}=\frac{D_x}{D}ww_x-\left(\int_x Ddx+C\right)w_{xx}+\frac{D_x}{D}\left(\int_x Ddx+C\right)w_x \tag{16a}$$

Rearrange the above equation as,

$$\left(ww_x\right)_x-\frac{D_x}{D}ww_x=-\left(\int_x Ddx+C\right)w_{xx}+\frac{D_x}{D}\left(\int_x Ddx+C\right)w_x=\left(\int_x Ddx+C\right)A$$

and each solution is,

$$ww_x=D\left[\int_x\left(\int_x Ddx+C\right)\frac{A}{D}dx+C_1\right],$$

$$w_x=-D\left(\int_x\frac{A}{D}dx+C_2\right) \text{ and}$$

$$w=-\int_x D\left(\int_x\frac{A}{D}dx+C_2\right)dx+C_3 \tag{16b}$$

Equalize the results as,

$$\left[\int_x D\left(\int_x\frac{A}{D}dx+C_2\right)dx+C_3\right]\left[D\left(\int_x\frac{A}{D}dx+C_2\right)\right]=D\left[\int_x\left(\int_x Ddx+C\right)\frac{A}{D}dx+C_1\right]$$

Expanding r.h.s., the following equation is produced,

$$\left[\int_x D\left(\int_x\frac{A}{D}dx+C_2\right)dx+C_3\right]=\left(\int_x Ddx+C\right)\left(\int_x\frac{A}{D}dx+C_2\right)\left[\left(\int_x\frac{A}{D}dx+C_2\right)+1\right]^{-1} \tag{16c}$$

Differentiate once,

$$\frac{A}{D}=\frac{D}{\left(\int_x Ddx+C\right)}\left(\int_x\frac{A}{D}dx+C_2\right)^3+\frac{D}{\left(\int_x Ddx+C\right)}\left(\int_x\frac{A}{D}dx+C_2\right)^2 \tag{18d}$$

which is the polynomial ODE with third order nonlinearity.

The First Integral for the Equation with Third Order Nonlinearity

In this case, suppose that, $H=\int_x\frac{A}{D}dx+C_2$ and $I=H+d$, then equation (16d) is transformed into,

$$I_x=\frac{D}{\left(\int_x Ddx+C\right)}\left[\begin{array}{l}I^3+\left(3d+1\right)I^2+\left(3d^2+2d\right)I\\+d^3+d^2-\dfrac{\left(\int_x Ddx+C\right)}{D}d_x\end{array}\right] \tag{17a}$$

The step now is to reconfigure the polynomial term in (17a) as follows,

$$\left(d_1y+d_2\right)I^2+d_3I+\left(d_1I+d_2\right)d_4 \text{ or}$$

$$d_1I^3+d_2I^2+d_3I+d_1d_4I+d_2d_4$$
$$d_1I^3+d_2I^2+\left(d_3+d_1d_4\right)I+d_2d_4$$

which each coefficient is described by,

$$d_1 = \frac{D}{\left(\int_x Ddx + C\right)}, d_2 = \frac{(3d+1)D}{\left(\int_x Ddx + C\right)},$$

$$d_3 + d_1 d_4 = \frac{\left(3d^2 + 2d\right)D}{\left(\int_x Ddx + C\right)}, \tag{17b}$$

$$d_2 d_4 = \frac{\left(d^3 + d^2\right)D}{\left(\int_x Ddx + C\right)} - d_x$$

Evaluating (17b), the expression for d_3 and d_4 are,

$$d_3 = \frac{\left(3d^2 + 2d\right)D}{\int_x Ddx} + \frac{d_x}{(3d+1)} - \frac{\left(d^3 + d^2\right)D}{(3d+1)\int_x Ddx}$$

$$d_4 = \frac{\left(d^3 + d^2\right)}{(3d+1)} - \frac{\int_x Ddx}{D(3d+1)} d_x \tag{17c}$$

The step is now to solve the Riccati equation (17a), let, $I = \beta_2\beta_3$, the equation can be rearranged as,

$$\beta_3\beta_{2t} - \left(d_1 I + d_2\right)\beta_2^2\beta_3^2 - d_3\beta_2\beta_3 = -\beta_2\beta_{3t} + \left(d_1 I + d_2\right)d_4 = \gamma_3\beta_2\beta_3$$

and is separated as,

$$\beta_3\beta_{2t} - \left(d_1 I + d_2\right)\beta_2^2\beta_3^2 - \left(d_3 + \gamma_3\right)\beta_2\beta_3 = 0 \quad \text{and}$$
$$\beta_2\beta_{3t} + \gamma_3\beta_2\beta_3 - \left(d_1 I + d_2\right)d_4 = 0 \tag{18a}$$

The solutions for β_2 and β_3 are,

$$\beta_2 = e^{\int_x (d_3+\gamma_3)dx}\left[\int_x e^{\int_x (d_3+\gamma_3)dx}\left(d_1 I + d_2\right)\beta_3 dx + C_1\right]^{-1}$$

and

$$\beta_3 = e^{-\int_x \gamma_3 dx}\left[\int_x e^{\int_x \gamma_3 dx}\left(d_1 I + d_2\right)d_4 \frac{1}{\beta_2} dx + C_2\right] \tag{18b}$$

The relation for $I = \beta_2\beta_3$ is thus,

$$I = \beta_2\beta_3 = e^{\int_x d_3 dx}\left[\int_x e^{\int_x (d_3+\gamma_3)dx}\left(d_1 I + d_2\right)\beta_3 dx + C_1\right]^{-1}\left[\int_x e^{\int_x \gamma_3 dx}\left(d_1 I + d_2\right)d_4 \frac{1}{\beta_2} dx + C_2\right]$$

Without loss of generality, suppose that, $\beta_2 = \varphi e^{\int_x \gamma_3 dx}$, the above relation is performed as,

$$e^{\int_x d_3 dx}\left(d_1 I + d_2\right)\varphi\beta_2\beta_3\left[\int_x e^{\int_x d_3 dx}\left(d_1 I + d_2\right)\varphi\beta_2\beta_3 dx + C_1\right] = e^{2\int_x d_3 dx}\varphi\left(d_1 I + d_2\right)\left[\int_x \left(d_1 I + d_2\right)\frac{d_4}{\varphi} dx + C_2\right] \tag{18c}$$

Suppose that, $e^{2\int_x d_3 dx}\varphi = \frac{d_4}{\varphi}$, integrate the above equation to get,

$$\left[\int_x e^{\int_x d_3 dx}\left(d_1 I + d_2\right)\varphi I dx + C_1\right]^2 = \left[\int_x \left(d_1 I + d_2\right)\frac{d_4}{\varphi} dx + C_2\right]^2$$

Note that the integration constants are the same, in this case,

$$e^{\int_x d_3 dx}\varphi^2\left(d_1 I^2 + d_2 I\right) = \left(d_1 d_4 I + d_2 d_4\right) \text{ or}$$

$$\left(d_1 y^2 + d_2 y\right) = e^{\int_x d_3 dx}\left(d_1 y + d_2\right) \text{ or}$$

$$I^2 + \left(\frac{d_2}{d_1} - e^{\int_x d_3 dx}\right)I - e^{\int_x d_3 dx}\frac{d_2}{d_1} = 0 \tag{19a}$$

The roots are then,

$$I = e^{\int_x d_3 dx}, -\frac{d_2}{d_1} \tag{19b}$$

The solution for v is then reduced into the solution of the polynomial equation. Take the root as,

$$H = \int_x \frac{A}{D} dx + C_2 = I + d = e^{\int_x d_3 dx} + d \text{ or}$$

$$\int_x \frac{A}{D} dx + C_2 = e^{\int_x \left[\frac{\left(3d^2+2d\right)D}{\int_x Ddx} + \frac{d_x}{(3d+1)} - \frac{\left(d^3+d^2\right)D}{(3d+1)\int_x Ddx}\right]dx} + d$$

and

$$V = \exp\left\{\frac{D}{De^{\int_x \left[\frac{\left(3d^2+2d\right)D}{\int_x Ddx} + \frac{d_x}{(3d+1)} - \frac{\left(d^3+d^2\right)D}{(3d+1)\int_x Ddx}\right]dx} + d}\right\} \tag{19d}$$

From the relation in (15) and (16), the integral is then,

$$\int_x Ddx = \int_x \ln V dV = V\ln V - V \quad (20a)$$

Let, $B = e^{\int_x \left[\frac{(3d^2+2d)D}{\int_x Ddx} + \frac{d_x}{(3d+1)} - \frac{(d^3+d^2)D}{(3d+1)\int_x Ddx}\right]dx}$, then

$$\int_x Ddx = \frac{D}{DB+d} e^{\frac{D}{DB+d}} - e^{\frac{D}{DB+d}}$$

Rearranging the above expression,

$$e^{-\frac{D}{DB+d}} \int_x Ddx = \frac{D}{DB+d} - 1 \text{ or}$$

$$\left\{\left[\frac{(3d^2+2d)D^2}{\int_x Ddx} + \frac{d_x D}{(3d+1)} - \frac{(d^3+d^2)D^2}{(3d+1)\int_x Ddx} - D_x D + D_x\right]B - d_x - D_x d\right\}\left(D^2 - 2D^2B - 2Dd + D^2B^2 + 2DdB + d^2\right) =$$
$$(D_x D - D_x DB - D_x d)\left(D^3B^3 + 3dD^2B^2 + 3d^2DB + d^3\right) - (D_x B - d_x)\left(D^4B^4 + 4dD^3B^3 + 6d^2D^2B^2 + 4d^3DB + d^4\right)$$
$$-D\left[\frac{(3d^2+2d)D}{\int_x Ddx} + \frac{d_x}{(3d+1)} - \frac{(d^3+d^2)D}{(3d+1)\int_x Ddx}\right]B\left(D^4B^4 + 4dD^3B^3 + 6d^2D^2B^2 + 4d^3DB + d^4\right) \quad (20b)$$

which is fifth order polynomial equation in B. The following section is the proposed method to reduce the order of polynomial equation higher than four.

REDUCTION OF FIFTH ORDER POLYNOMIAL

Consider equation (9c) as follows,

$$d_1B^5 + d_2B^4 + d_3B^3 + d_4B^2 + d_5B + d_6 = 0$$

Multiply by the function, b and rearrange,

$$d_1E^5 + d_2\beta E^4 + d_3\beta^2E^3 + d_4\beta^3E^2$$
$$+d_5\beta^4E + d_6\beta^5 + \varphi = \varphi \quad (21a)$$

where, $B = \beta A$. Rearranged equation (21a) as given by,

$$\left(b_1E^3 + b_2E^2 + b_3E + b_4\right)E^2 + b_5E^2$$
$$-\left(b_1E^3 + b_2E^2 + b_3E + b_4\right)b_6 = \varphi \quad (21b)$$

Expanding the all the coefficients as,

$$b_1E^5 + b_2E^4 + (b_3 - b_1b_6)E^3 + (b_4 + b_5 - b_2b_6)E^2$$
$$-b_3b_6E - b_4b_6 = \varphi$$

Relate the coefficients as in the following,

$$b_1 = d_1$$
$$b_2 = d_2\beta$$
$$b_3 - b_1b_6 = d_3\beta^2$$
$$b_4 + b_5 - d_2\beta\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) = d_4\beta^3$$
$$-\frac{1}{d_1^2}\left(b_3 - d_3\beta^2\right)^2 = d_5\beta^4 + d_3\beta^2\frac{1}{d_1}\left(b_3 - d_3\beta^2\right)$$
$$-\left[d_4\beta^3 + d_2\beta\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) - b_5\right]\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) = d_6\beta^5 + \varphi$$
$$\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) = -\left\{\frac{\left[d_4\beta^3 + d_2\beta\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) - b_5\right]\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) + d_5\beta^5}{b_5}\right\} \quad (21c)$$

The fifth equation of (21c) gives the roots as,

$$\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) = \frac{1}{2}\beta^2\left[d_3 \pm \left(d_3^2 - 4d_5\right)^{\frac{1}{2}}\right] = \beta^2 f_0 \qquad (21d)$$

Moving to the last equation, the functions b_5 and β are disappear from the operation. In this case we will consider the test function, $b_5 + \beta^{10} = d_6\beta^5$, and will perform as,

$$\beta^{10} + d_4\beta^3\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) + d_2\beta\frac{1}{d_1^2}\left(b_3 - d_3\beta^2\right)^2 + b_5 = 0 \qquad (22a)$$

Substituting for b_3, the expression for β is,

$$\beta^{10} + \left(d_4 f_0 + d_2 f_0^2\right)d_4\beta^5 + b_5 = 0$$

$$\beta^5 = -\frac{1}{2}\left(d_4 f_0 + d_2 f_0^2\right) \pm \frac{1}{2}\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right]^{\frac{1}{2}} \qquad (22b)$$

Substitute back to, $b_5 + \beta^{10} = d_6\beta^5$ as follows,

$$b_5 = \frac{1}{4}\left(d_4 f_0 + d_2 f_0^2\right)^2 - \frac{1}{4}\left[\frac{\left(d_4 f_0 + d_2 f_0^2\right)^2 + d_6\left(d_4 f_0 + d_2 f_0^2\right)}{\left(d_4 f_0 + d_2 f_0^2 + d_6\right)}\right]^2 \qquad (22c)$$

which is then, solves b_5, β, φ thus generates all the coefficients of b_i. The polynomial is then rewritten as,

$$\left(b_1E^3 + b_2E^2 + b_3E + b_4\right)\left(E^2 - b_6\right) = -b_5\left(E^2 - \frac{\varphi}{b_5}\right)$$

which is reduced as,

$$d_1B^3 + d_2B^2 + \left\{\frac{1}{2}d_1\left[d_3 \pm \left(d_3^2 - 4d_5\right)^{\frac{1}{2}}\right] + d_3\right\}B + d_4 + \frac{1}{2}d_2\left[d_3 \pm \left(d_3^2 - 4d_5\right)^{\frac{1}{2}}\right] = 0$$

Note that the reduction procedure will solve (19d), perform the log operation and differentiate once, the higher polynomial equation for $G = \int_x Ddx$ is produced. Applying the reduction procedure once more thus the result of integral evaluation is explicitly obtained.

CONCLUSION

The solvability of the elastic wave equation in homogeneous media is analyzed in this research. The spatial coordinates of the coupled wave equations are eliminated by Fourier transformation in the whole and then half domains. The order reductions for ODEs and polynomial equation are proposed to produce the complete solutions in time and spatial domain. The reduction is performed by separating the ODEs coefficients iteratively until the second order is obtained which produces complex wave propagation. It is found that the evaluation of integral is possible by solving the third order Riccati equation and reducing the higher order polynomials.

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DECLARATION OF COMPETING INTEREST

None

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