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Solving SEI Model using Non-Standard Finite Difference and High Order Extrapolation with Variable Step Length

(Algoritma Panjang Langkah Boleh Ubah Dengan Skim Perbezaan Terhingga Bukan Standard yang Diekstrapolasi Tertib Tinggi Untuk Model SEI)

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ABSTRACT

A high-level method was obtained to solve the SEI model problem involving Symmetrization measures in numerical calculations through the Implicit Midpoint Rule method (IMR). It is obtained using Non-Standard Finite Difference Schemes (NSFD) with Extrapolation techniques combined. In solving differential equation problems numerically, the Extrapolated SEI model method is able to generate more accurate results than the existing numerical method of SEI model. This study aims to investigate the accuracy and efficiency of computing between Extrapolated One-Step Active Symmetry Implicit Midpoint Rule method (1ASIMR), Extrapolated One-Step Active Symmetry Implicit Midpoint Rule method (2ASIMR), Extrapolated One-Step Passive Symmetry Midpoint Rule method (1PSIMR) and the extrapolated Two-Step Passive Symmetry Midpoint Rule method (2PSIMR). The results show that the 1ASIMR method is the most accurate method. For the determination of the efficiency of 2ASIMR and 2PSIMR methods have high efficiency. At the end of the study, the results from the numerical method obtained show that Extrapolation using Non-Standard Finite Difference has higher accuracy than the existing Implicit Midpoint Rule method.

Keywords: Non-standard finite difference schemes;Extrapolation; SEI model; Implicit midpoint rule; Symmetrization

ABSTRAK

Kaedah peringkat tinggi diperoleh untuk menyelesaikan masalah model SEI yang melibatkan langkah simetri (symmetrization) dalam pengiraan berangka melalui kaedah kaedah titik tengah tersirat (IMR). Ia diperoleh menggunakan teknik ekstrapolasi yang digabungkan dengan skema perbezaan terhingga bukan standard. Dalam menyelesaikan masalah persamaan kebezaan secara berangka, kaedah model SEI yang diekstrapolasi mampu menjana keputusan yang lebih tepat berbanding kaedah berangka model SEI yang sedia ada. Hal ini bertujuan untuk menyiasat ketepatan dan kecekapan pengkomputeran antara kaedah titik tengah tersirat simetri aktif satu langkah yang diekstrapolasi(1ASIMR), kaedah titik tengah tersirat simetri aktif dua Langkah yang diekstrapolasi(2ASIMR), kaedah titik tengah simetri pasif satu langkah yang diekstrapolasi(1PSIMR) dan kaedah titik tengah simetri pasif dua langkah yang diekstrapolasi(2PSIMR). Hasil kajian menunjukkan bahawa kaedah 1ASIMR adalah kaedah yang paling tepat. Bagi penentuan kecekapan kaedah kaedah 2ASIMR dan 2PSIMR mempunyai kecekapan yang tinggi. Pada akhir kajian, keputusan daripada kaedah berangka yang diperoleh menunjukkan bahawa skim perbezaan terhingga bukan standard ekstrapolasi mempunyai ketepatan yang lebih tinggi berbanding kaedah berangka yang sedia ada.

Kata kunci: Perbezaan terhingga bukan standard; ekstrapolasi; Model SEI; kaedah titik tengah; tersimetri

INTRODUCTION

Malware propagation is being investigated extensively because it poses a significant danger to information exchange. It also represents one of the most critical security threats we must confront. As with a biological disease, latent period can be infected from the spread of the virus. Same goes to the mode of susceptibility, transmission, and resistance (Hethcote 2000), also the device itself, the spatial structure, and the environment in which it could be disseminated with other factors (Akmam et al. 2021).

The theory systems of Ordinary Differential Equations (ODEs) are being used to explore the dynamic spread of the virus by using the mathematical models. The standard epidemic models that can describe the spread of the virus is susceptible (S), exposed (E), infected (I) and recovered (R) (Androulidakis et al. 2016).

Numerical methods are used to solve initial value problems in differential equations are divided into two categories namely Multi-Step Method and Runge-Kutta Method. Each of these numerical techniques is often used in solving Ordinary Differential Equation (ODE) problems. To solve closely related methods such as the Implicit Midpoint of Symmetry Method, the numerical solution must be near to zero in order to get the exact solution. It applies to methods that combine the same features for both the Multi-Step Method and the Runge-Kutta Method (Butcher 1966). There are various existing methods that involve numerical methods to solve initial value problems. The two main methods, namely One Step method such as Euler method and Implicit Midpoint and multi-Step method such as Adams-Moulton and Euler-Romberg method (Atkinson et al. 2011). The One Step method is easy to use and it divides the solution by a segment of a short straight line. The Implied Midpoint (IMR) method is extended to Two Step Symmetry. This symmetry is equivalent to a Symmetrization applied in two steps. The advantage of Two-Step Active symmetry, the method has a sequence of two behaviors compared to a sequence of One Step symmetry behaviors (Razali & Chan 2015).

The main method that involves numerical methods to solve initial value problems in this system is called the Symmetrized Implicit Midpoint Rule (IMR). It has one step and two step methods, both of this step have active and passive modes, which are One Step Passive Method of Symmetry (N. Razali et al. 2018), One Step Active Method of Symmetry, Two Step Active Method of Symmetry and Two Step Passive Method of Symmetry (1PSIMR,1ASIMR,2ASIMR,2PSIMR). All of this method is Extrapolated in order to obtain higher order method (Gorgey 2018). It also has the advantage suppressing the order reduction when used with higher order methods.

METHODOLOGY

The flow of numerical experiment can be summarized in the flowchart as shown in figure 1. Problem from SEI model is chosen to test the performance studied of Extrapolated Implicit Midpoint Rule methods, namely 1ASIMR, 1PSIMR, 2ASIMR and 2PSIMR. The coding for each method is establish in MATLAB program and is checked for errors. The selected benchmark problem is then added into the program for computation after verification stage. For each approach under investigation, the computing processes are repeated. To uncover issues in the codes, the resulting results are verified for consistency. If the results are normal, the investigation process for the performance of 1ASIMR, 1PSIMR, 2ASIMR and 2PSIMR in solving the benchmark problem is carried out. The approach is to calculate the average and computation time for each method, which will be detailed in the Results and Discussion.

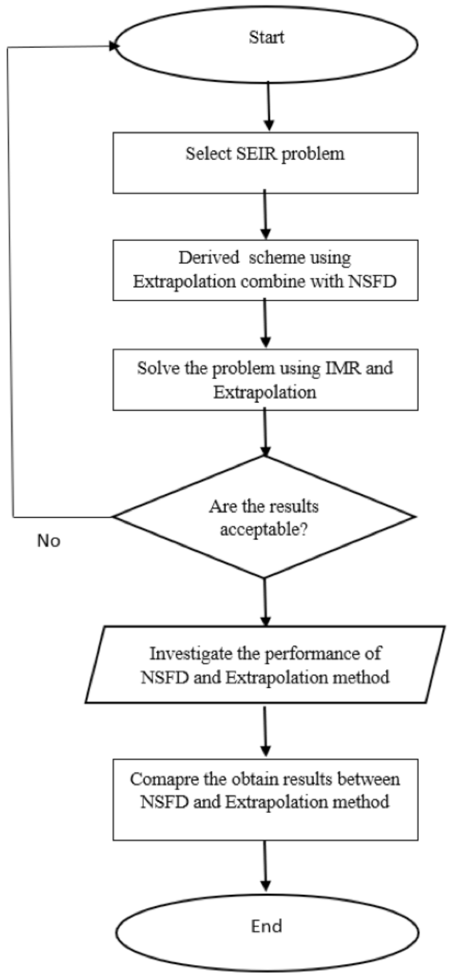

FIGURE 1. Flow chart of research

Comparison of the numerical performance between 1ASIMR, 1PSIMR, 2ASIMR and 2PSIMR is performed and relevant conclusion is drawn.

SEI MODEL

$$\frac{dS}{dT} = \mu(N - S) - \beta SI, \qquad S(0) = S_0$$
$$\frac{dE}{dt} = \beta SI - (\mu + \sigma)E, \qquad E(0) = E_0 \tag{1}$$
$$\frac{dI}{dt} = \sigma E - (\mu + \gamma)I, \qquad I(0) = I_0$$

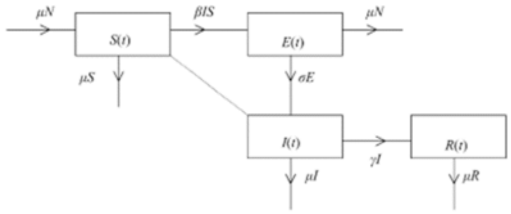

FIGURE 2. compartment diagram for SEIR model

$$S^{n+1} = \frac{S^n + h_n \mu N}{1 + h_n(\mu + \beta I^n)},$$
$$E^{n+1} = \frac{E^n + h_n \beta I^n S^{n+1}}{1 + h_n(\mu + \sigma)}, \quad (2)$$
$$I^{n+1} = \frac{I^n + h_n \sigma E^{n+1}}{1 + h_n(\mu + \gamma)}$$

The proposed approaches' fixed points will be the same as the critical point of the SEI model and they will have the same stability properties. Richardson's Extrapolation is a well-known approach for obtaining higher-order procedures, and it has been much discussed in the scientific literature (Hairer & Norsett 1993). Firstly, we compute the numerical outcomes of the initial value problem using a numerical approach of order p (1) by performing n_i steps with step size h_i to obtain $yh_i(x_0 + h) := T_{i,1}$. Then we calculate for $h_1 > h_2 > h_3 > \ldots$ (taking $h_i = h/n_i$, with n_i as a positive integer) (Xiong et al. 2020).

DERIVING EXTRAPOLATION WITH NSFD

"Harmonic sequence" has been used in this equation to be considered in Richardson Extrapolation. This method is used because easy to calculate the number of methods increasing by using the algorithm of Aitken-Neville:

$$T_{j,k+1} = T_{j,k} + \frac{(j-k)(T_{j,k} - T_{j-1,k})}{k} \quad (3)$$

the second-order methods:

$$y(\widehat{x_0 + h}) = T_{2,2} = 2T_{2,1} - T_{1,1} = 2y_{\frac{h}{2}}(x_0 + h) - y_h(x_0 + h), \quad (4)$$

the third-order methods:

$$y(\widehat{x_0 + h}) = T_{3,3} = \frac{9y_{\frac{h}{3}}(x_0 + h) - 8y_{\frac{h}{2}}(x_0 + h) + y_h(x_0 + h)}{2}, \quad (5)$$

Lastly, fourth-order methods:

$$y(\widehat{x_0 + h}) = T_{4,4} = \frac{-T_{1,1} + 24T_{2,1} - 81T_{3,1} + 64T_{4,1}}{6} = \frac{64y_{\frac{h}{4}}(x_0 + h) - 81y_{\frac{h}{3}} + 24y_{\frac{h}{2}}(x_0 + h) - y_h(x_0 + h)}{6}. \quad (6)$$

The constant β will be used in the MATLAB to obtain the result for accuracy and efficiency of the SEI model (Jansen & Twizell 2002) $S = N$, by setting all the derivative equal to zero. Which is $E = 0$, $I = 0$ (Martín-Vaquero et al. 2018).

$$S^* = \frac{(\mu + \sigma)(\mu + \gamma)}{\sigma\beta},$$
$$E^* = \frac{\mu N}{\mu + \sigma} - \frac{\mu(\mu + \gamma)}{\sigma\beta}, \quad (7)$$
$$I^* = \frac{\mu\sigma N}{(\mu + \sigma)(\mu + \gamma)} - \frac{\mu}{\beta},$$

while the continuous model's bifurcation point (1) is

$$\beta^* = \frac{(\mu + \sigma)(\mu + \gamma)}{\sigma N} \quad (8)$$

TABLE 1. Stability of critical point in SEI model

β \ critical point	trivial	not-trivial
$< \beta$	stable	Not stable
$= \beta$	Naturaly not stable	-
$> \beta$	Not stable	stable

VARIABLE-STEP ALGORITHMS

For successful integration of these models, adaptive step size selection is critical.

$$h_{n+1} = h_n min\left(facmax, max\left(facmin, fac \cdot (1/err)^{1/k}\right)\right) \quad (9)$$

The *fac* =0.5 is the number of factors use, *facmax*=4 is the highest number of factors to use and *facmin*=10^{-1} is the lowest number of factors to use while k is the number of orders for Extrapolation.

The error is used to describe the fast of the length step whether increase or decrease. By calculate with $T_{k,k} - T_{k,k-1}$, we can estimate the error and we can calculate the tolerance by *sc*. The value of *err* and *sc* can be calculated as:

$$err = \sqrt{\frac{1}{3}\sum_{i=1}^{3} \left(\frac{(T_{k,k} - T_{k,k-1})_i}{sc_i}\right)^2} \quad (10)$$

$$sc_i = ATol_i = max(|y_{0,i}|, |T_{k,k,i}|) \cdot RTol_i, \quad (11)$$

where $y_{0,i}$ is the number of component from the previous step length, and $T_{k,k,i}$ is the number of component that been obtained from the population.

$$h_{n+1} = h_n min \left(facmax, max \left(facmin, fa \cdot \frac{err_n^{\alpha}}{err_{n+1}^{1/k}} \right) \right) \quad (12)$$

were α = 0.08 (when the previous step is accepted, if it is rejected α = 0) as it was suggested in (Gustafsson 1991) (Hairer & Norsett 1993).

The sets of parameters and are very important to detect the efficiency of computation because it can be compared in the numerical solution.

RESULT AND DISCUSSION

ORDER BEHAVIOUR OF SYMMETRIZED IMPLICIT MIDPOINT RULE

In order to obtain a good stability analysis in a Numerical Analysis is by focusing on the behavior of the global error (t). this is because the behavior can be determined by the ODE solution which are the stability depends on the step size (Al-Mutib 1984). Figure 3 show the order behavior of the Symmetrized of a SEI model.

TABLE 2. Definition of parameter

Variable	Parameter value	Definition
N	50000000	Human population
μ	0.02	Rate of natural birth
m	8	Probability of changing from E to I
g	5	Rate of recovery
β	0.0000005	Transmission rate
S	1250000	Suspected population
E	50000	Exxposed population
I	30000	Infected population

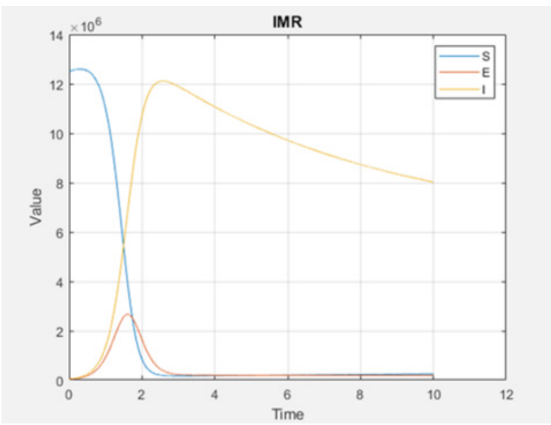

FIGURE 3. Behaviour of compartment population over time for SEI model

EFFICIENCY OF SYMMETRIZATION IMPLICIT MIDPOINT RULE

The Numerical method's efficiency can be measured by using the CPU time. It can solve a problem to a certain level of precision. The total number of iterations and the CPU time per iteration determine the efficient of the model. The result can show the numerical algorithm by producing the behavior of the graph. The stiffness Symmetrization IMR's efficiency is comparable to that of the dynamic analysis approach in terms of total CPU time (Lewis 1989).

Figure 4 to Figure 6 show the maximum global fault diagram against the sum of the functions of this evaluator plotted to determine the efficiency of the Implicit Midpoint Rule of One-Step and Two-Step Symmetrization. From the results obtained, 2ASIMR have highest efficiency.

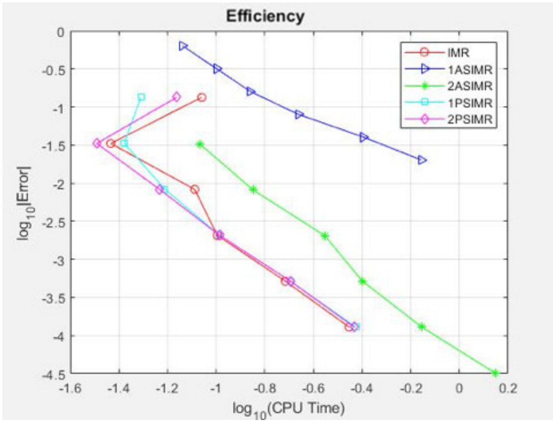

FIGURE 4. Efficiency diagram for β=0.0000005

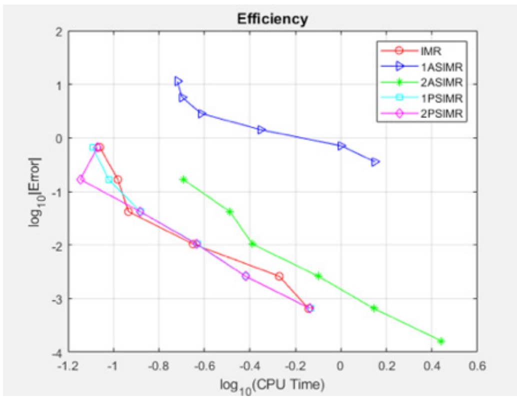

FIGURE 5. Efficiency diagram for β=0.000005

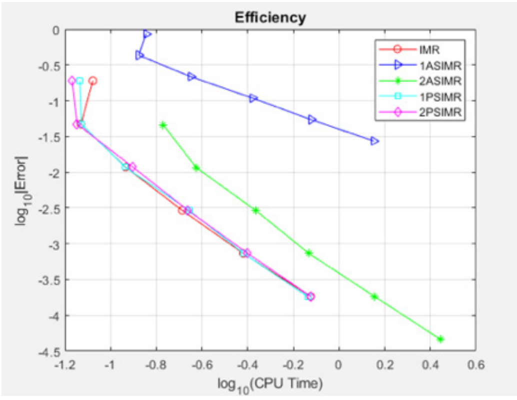

FIGURE 6. Efficiency diagram for β=0.00005

ACCURACY OF SYMMETRIZED IMPLICIT MIDPOINT RULE

The Symmetrized Implicit Midpoint Rule analysis algorithm evaluates the accuracy of a numerical solution by detecting the solution of the problem as the original problem similarity. This help to improve the Symmetrized Implicit Midpoint Rule analysis from creating a lot of error. The rise of the bound can be observed to detect error analysis which most cases far exceed the Numerical error values themselves (Rogalev & Rogalev 2020). Figure 7 to Figure 9 show that 2ASIMR have the most accurate compare to other methods.

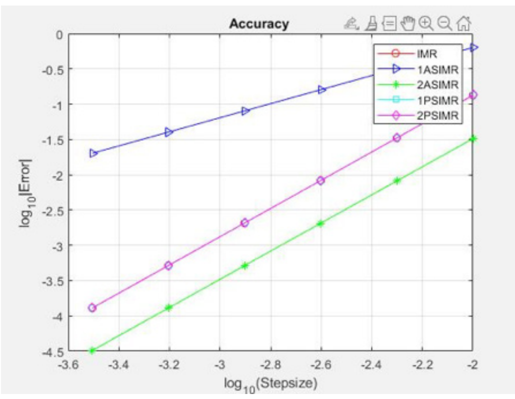

FIGURE 7. Accuracy diagram for β=0.0000005

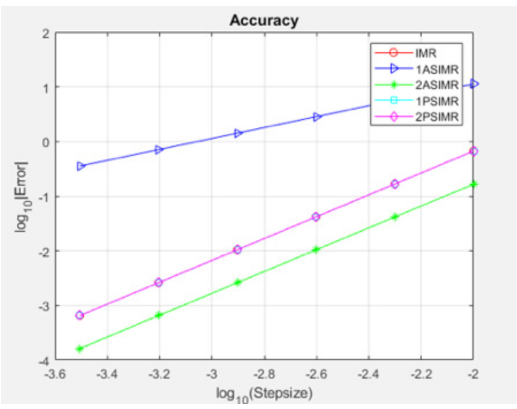

FIGURE 8. Accuracy diagram for β=0.000005

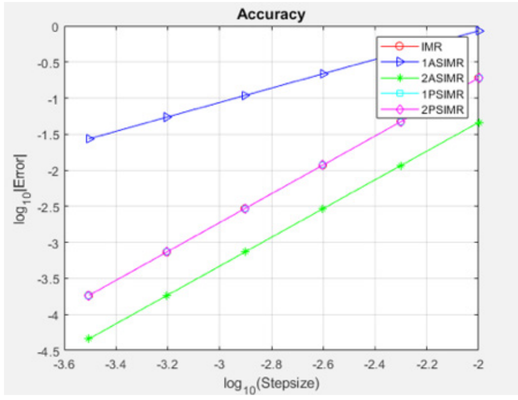

FIGURE 9. Accuracy diagram for β=0.00005

CONCLUSION

To solve the SEI model for malware spread, high-order approaches are used with the Non-Standard Different Scheme (NSFD) that was discovered throughout the research. As a result, the new method produces result that dynamically consistent and stable with the SEI model.

The analysis of Numerical show a smooth solution, but the model is constantly changing and some variables are changing at the same time. As a result, addressing the solution problem is critical in changing the step size. In discrete schemes, different methods have been tried to control the error. Finally, these adaptive step sizes were compared to determine the approaches' acceleration and efficiency.

Finally, the computational accuracy of 1PSIMR, 2ASIMR and 2PSIMR is lower than 1ASIMR where the accuracy increase through the time passes. For the determination of the Efficiency, 2ASIMR and 2PSIMR methods are the most efficient due to the extra symmetrization step (Zlatev et al. 2017). However, due to the fact that 1ASIMR is more accurate that 2ASIMR and 2PSIMR at large space and time steps at which the computation time is short. At the end of the study, the results from the numerical method obtained show that the non-standard finite difference scheme of Extrapolation has higher accuracy than the existing numerical method.

The idea of merging Extrapolation approaches with other non-standard finite difference schemes for different types of issues could be studied in the future. This can be accomplished by using a higher order approach with multiple parameters to solve the autonomous differentiation system, which is a challenge in the traditional scheme.

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