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NON-REGULAR QUADRATIC STOCHASTIC OPERATORS WITH COUNTABLE STATE SPACE

(Ketidaksekataan Operator Stokastik Kuadratik dengan Set Terbilang)

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ABSTRACT

In this paper we present an example of non-regular extremal Volterra quadratic stochastic operator defined on infinite-dimensional simplex. We show that the construction of this operator to odd dimensional face of infinite-dimensional simplex is non-regular transformation. We employ method of approximation to prove non-regularity of initial operator.

Keywords: ergodic hypothesis; regular transformation; extremal Volterra quadratic stochastic operator with infinite state space.

ABSTRAK

Dalam kertas kerja ini, kami memperkenalkan satu contoh operator stokastik kuadratik Volterra yang ekstrim pada simplex berdemensi tak terhingga. Kami menunjukkan bahawa pembentukan operator ini pada muka dimensi ganjil daripada simpleks dimensi tak terhingga adaladah penjelmaan yang tidak sekata. Kami menggunakan kaedah penghampiran untuk membuktikan ketidaksekataan pada operator awal.

Kata kunci: hipotesis ergodik; transfomasi sekata; operator stokastik kuadratik Volterra yang ekstrim pada set tak terhingga.

1. Introduction

Let $(X, \mathcal{F})$ be a measurable space, where X is a state space and $\mathcal{F}$ is σ-algebra on X, and $S(X, \mathcal{F})$ be the set of all probability measures on $(X, \mathcal{F})$. Let $\{P(x, y, A) : x, y \in X, A \in \mathcal{F}\}$ be a family of functions on $X \times X \times \mathcal{F}$ such that $P(x, y, \cdot) \in S(X, \mathcal{F})$; $P(x, y, A)$ is a measurable function on $(X \times X, \mathcal{F} \otimes \mathcal{F})$ which regarded as a function of two variables of x and y with fixed $A \in \mathcal{F}$; and $P(x, y, A) = P(y, x, A)$ for any $x, y \in X$ and $A \in \mathcal{F}$.

By specifying such collection of functions $\{P(x, y, A) : x, y \in X, A \in \mathcal{F}\}$ we introduce a non-linear transformation $V : S(X, \mathcal{F}) \to S(X, \mathcal{F})$ which defined by

$$(V\lambda)(A) = \int_X \int_X P(x, y, A) d\lambda(x) d\lambda(y), \tag{1}$$

where $A \in \mathcal{F}$ is an arbitrary measurable set.

The case with finite state space X was considered by Bernstein (1942). If $X = \{1, 2, \cdots, m\}$, then a set of all probability measures on X is the finite-dimensional simplex $S^{m-1} = \{\mathbf{x} = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m : \forall i \ x_i \geq 0, \sum_{i=1}^m x_i = 1\}$; a family of functions $\{P(x, y, A) : x, y \in X, A \in \mathcal{F}\}$ one can represent as a cubic matrix $(P_{ij,k})_{i,j,k=1}^m$ with

$$a) P_{ij,k} \geq 0; \ \ b) P_{ij,k} = P_{ji,k}; \ \ c) \sum_{k=1}^m P_{ij,k} = 1 \text{ for all } i, j, k \in X,$$

where $P_{ij,k} = P(ij, \{k\})$. Such transformation V is called quadratic stochastic operator (QSO). When the state space $X = \mathbb{N}$ is an infinite countable set of positive integers, and $\mathcal{F}$ is the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$, i.e. the set of all subsets of $\mathbb{N}$, then

$$S^{\mathbb{N}} = \{\mathbf{x} = (x_i)_{i=1}^{\infty} : \forall i \;\; x_i \geq 0, \sum_{i=1}^{\infty} x_i = 1\} \tag{2}$$

is the set of all probability measures on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

Definition 1.1. The set $S^{\mathbb{N}}$ is called infinite-dimensional simplex.

In this case one can specify a family of functions $\{P(i, j, \{k\}) : i, j \in \mathbb{N}, \{k\} \in \mathcal{P}(\mathbb{N})\}$ as a tensor $\{p_{ij,k} : i, j, k \in \mathbb{N}\}$, and by additivity $P(i, j, A) = \sum_{k \in A} p_{ij,k}$ for arbitrary $A \in \mathcal{P}(\mathbb{N})$. Then $P(i, j, \mathbb{N}) = \sum_{k=1}^{\infty} p_{ij,k} = 1$. In this case, the measure $P(i, j, \cdot)$ is a discrete probability measure and the corresponding QSO $V : S^{\mathbb{N}} \to S^{\mathbb{N}}$ has the following form

$$(V\mathbf{x})_k = \sum_{i,j=1}^{\infty} P_{ij,k} x_i x_j, \;\; k \in \mathbb{N} \tag{3}$$

where the coefficients $P_{ij,k}$ satisfy the following conditions:

$$a) P_{ij,k} \geq 0; \;\; b) P_{ij,k} = P_{ji,k}; \;\; c) \sum_{k=1}^{\infty} P_{ij,k} = 1 \text{ for all } i, j, k \in \mathbb{N}.$$

2. Infinite Dimensional Simplex

We consider infinite-dimensional simplex $S^{\mathbb{N}}$. Let $intS^{\mathbb{N}}$ denote the interior and $\partial S^{\mathbb{N}}$ denote the boundary of $S^{\mathbb{N}}$, where

$$intS^{\mathbb{N}} = \{\mathbf{x} \in S^{\mathbb{N}} : \forall i \;\; x_i > 0\}, \text{ and } \quad \partial S^{\mathbb{N}} = S^{\mathbb{N}} \setminus intS^{\mathbb{N}}.$$

For arbitrary subset $\alpha \subset \mathbb{N}$ the $\Gamma_\alpha = \{x \in S^{\mathbb{N}} : x_i = 0, \;\; i \notin \alpha\}$ is called the face of infinite-dimensional simplex $S^{\mathbb{N}}$. It is evident that a face Γ_α is the simplex.
If $\alpha = \{i\}$ consists of a single point i, then a face Γ_i is the ith vertex $M_i = (\delta_{1i}, \delta_{2i}, \cdots, \delta_{mi}, \cdots)$ of the simplex $S^{\mathbb{N}}$, where δ_{ij} is the Kronecker symbol.

One can consider two type faces: finite-dimensional face Γ_α if α is a finite subset, and respectively infinite-dimensional face if α is infinite subset. We select a family of finite-dimensional faces Γ_{Λ_k} such that $\Lambda_k = \{1, 2, \cdots, 2k+1\}$ where $k \in \mathbb{N}$. It is evident $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots \subset \Lambda_n \subset \cdots$ and $\cup_{k=1}^{\infty} \Lambda_k = \mathbb{N}$.

3. Volterra Quadratic Stochastic Operators with Countable State Space

The quadratic stochastic operator V (3) is called Volterra, if $p_{ij,k} = 0$ for any $k \notin \{i, j\}$. The biological treatment of such operators is rather clear: the offspring repeats one of its parents. Evidently for any Volterra QSO

$$p_{ii,i} = 1; \;\; p_{ik,k} + p_{ki,i} = p_{ik,k} + p_{ik,i} = 1 \text{ for all } i, k \in \mathbb{N}, i \neq k. \tag{4}$$

A Volterra QSO V defined on $S^{\mathbb{N}}$ has the following form

$$(V\mathbf{x})_k = x_k^2 + 2\sum_{i=1, i\neq k}^{\infty} p_{ik,k}x_i x_k, \tag{5}$$

where $k \in \mathbb{N}$.

A QSO V is a Volterra if and only if

$$(V\mathbf{x})_k = x_k(1 + \sum_{i=1}^{\infty} a_{ki}x_i) \tag{6}$$

where $A = (a_{ij})_1^{\infty}$ is a skew-symmetric matrix with $a_{ki} = 2p_{ik,k} - 1$ for $i \neq k$, $a_{ii} = 0$ and $|a_{ij}| \leq 1$. Here $i, j \in \mathbb{N}$.

Suppose $\{V^k(\mathbf{x}) \in S^{\mathbb{N}} : k = 0, 1, 2, \cdots\}$ is a trajectory of the initial point $\mathbf{x} \in S^{\mathbb{N}}$, where $V^{k+1}(\mathbf{x}) = V(V^k(\mathbf{x}))$ for all $k = 0, 1, 2, \cdots$, with $V^0(\mathbf{x}) = \mathbf{x}$.

In this paper we will study a trajectory behavior of Volterra operators with countable state space. A point $\mathbf{a} \in S^{\mathbb{N}}$ is called a fixed point of a QSO V if $V(\mathbf{a}) = \mathbf{a}$. Let $Fix(V)$ be a set of all fixed points of QSO V. It is evident that for any Volterra QSO any vertex M_i is the fixed point of QSO V, i.e. $\{\mathbf{M}_1, \mathbf{M}_2, \cdots\} \subset Fix(V)$. Also it is evident that any face Γ_α is invariant subset of Volterra QSO V, that is $V(\Gamma_\alpha) \subset \Gamma_\alpha$. Thus if the face Γ_α is a finite-dimensional simplex, then QSO V is reduced to QSO $V_\alpha : \Gamma_\alpha \to \Gamma_\alpha$.

A QSO V is called a regular if for any initial point $\mathbf{x} \in S^{\mathbb{N}}$, a limit

$$\lim_{n\to\infty} V^n(\mathbf{x}) \tag{7}$$

exists.

Note that the limit point is a fixed point of a QSO V. Thus the fixed points of QSO describe a limit or a long run behavior the trajectories at any initial point. A limit behavior of trajectories and the fixed points of QSO play an important role in many applied problems (Ganikhodjaev 1993; Jenks 1969; Kesten 1970; Lyubich 1992; Lyubich 1971).

4. Ergodicity of QSO

In statistical mechanics an ergodic hypothesis proposes a connection between dynamics and statistics. In the classical theory the assumption was made that the average time spent in any region of phase space is proportional to the volume of the region in terms of the invariant measure, more generally, that time averages may be replaced by space averages. For quadratic stochastic operators Ulam (1960) suggested an analogue of a measure-theoretic ergodicity, the following ergodic hypothesis: A nonlinear operator V defined on the finite-dimensional simplex S^{m-1} is called ergodic if the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} V^k(x) \tag{8}$$

exists for any $\mathbf{x} \in S^{m-1}$. It is evident that a regular QSO V is ergodic; however, regularity does not follow from the ergodicity

On the basis of numerical calculations, Ulam (1960) conjectured that an ergodic theorem holds for any QSO V. In 1977, Zakharevich (1978) proved that in general this conjecture is false. He considered the following Volterra operator on S^2

$$\begin{aligned} x_1' &= x_1(1 + x_2 - x_3) \\ x_2' &= x_2(1 - x_1 + x_3) \\ x_3' &= x_3(1 + x_1 - x_2) \end{aligned} \tag{9}$$

and proved that it is a non-ergodic transformation. Later in Ganikhodjaev and Zanin (2004), the authors established a necessary and sufficient condition to be non-ergodic transformation for QSO defined on S^2. Recently in Ganikhodjaev *et al.* (2013), the authors studied special classes of a Volterra QSO defined on S^3. While the infinite dimensional Volterra operators are discussed in Mukhamedov *et al.* (2005), Mukhamedov *et al.* (2020) and Embong and Mukhamedov (2023).

5. Generalization of Zakharevich's Example on Infinite-Dimensional Simplex

In this paper we present the generalization of Zakharevich's example (9) to operator with infinite state space as follows.

The following QSO are defined on $S^{\mathbb{N}}$ one can consider as a generalization of Zakharevich's QSO

$$\begin{aligned} (V\mathbf{x})_1 &= x_1[1 + x_2 - x_3 + x_4 - x_5 + x_6 - x_7 + \cdots] \\ (V\mathbf{x})_2 &= x_2[1 - x_1 + x_3 - x_4 + x_5 - x_6 + x_7 - \cdots] \\ &\dots \\ &\dots \\ (V\mathbf{x})_{2n-1} &= x_{2n-1}[1 + x_1 - x_2 + \cdots - x_{2n-2} + x_{2n} - x_{2n+1} + x_{2n+2} - \\ &\quad x_{2n+3} + \cdots] \\ (V\mathbf{x})_{2n} &= x_{2n}[1 - x_1 + x_2 - \cdots - x_{2n-1} + x_{2n+1} - x_{2n+2} + \cdots] \\ &\dots \\ &\dots \end{aligned} \tag{10}$$

On face Γ_{Λ_1} with $\Lambda_1 = \{1, 2, 3\}$ and $V_1 : \Gamma_{\Lambda_1} \to \Gamma_{\Lambda_1}$ we have

$$\begin{aligned} (V_1\mathbf{x})_1 &= x_1[1 + x_2 - x_3] \\ (V_1\mathbf{x})_2 &= x_2[1 - x_1 + x_3] \\ (V_1\mathbf{x})_3 &= x_3[1 + x_1 - x_2] \end{aligned} \tag{11}$$

or

$$\begin{aligned} (V_1\mathbf{x})_1 &= x_1^2 + 2x_1x_2 \\ (V_1\mathbf{x})_2 &= x_2^2 + 2x_2x_3 \\ (V_1\mathbf{x})_3 &= x_3^2 + 2x_3x_1 \end{aligned} \tag{12}$$

On face Γ_{Λ_2} with $\Lambda_2 = \{1, 2, 3, 4, 5\}$ and $V_2 : \Gamma_{\Lambda_2} \to \Gamma_{\Lambda_2}$ we have

$$\begin{aligned}
(V_2\mathbf{x})_1 &= x_1[1 + x_2 - x_3 + x_4 - x_5] \\
(V_2\mathbf{x})_2 &= x_2[1 - x_1 + x_3 - x_4 + x_5] \\
(V_2\mathbf{x})_3 &= x_3[1 + x_1 - x_2 + x_4 - x_5] \\
(V_2\mathbf{x})_4 &= x_4[1 - x_1 + x_2 - x_3 + x_5] \\
(V_2\mathbf{x})_5 &= x_5[1 + x_1 - x_2 + x_3 - x_4]
\end{aligned} \tag{13}$$

or

$$\begin{aligned}
(V_2\mathbf{x})_1 &= x_1^2 + 2x_1x_2 + 2x_1x_4 \\
(V_2\mathbf{x})_2 &= x_2^2 + 2x_2x_3 + 2x_2x_5 \\
(V_2\mathbf{x})_3 &= x_3^2 + 2x_3x_1 + 2x_3x_4 \\
(V_2\mathbf{x})_4 &= x_4^2 + 2x_4x_2 + 2x_4x_5 \\
(V_2\mathbf{x})_5 &= x_5^2 + 2x_5x_1 + 2x_5x_3
\end{aligned} \tag{14}$$

On face Γ_{Λ_3} with $\Lambda_3 = \{1, 2, 3, 4, 5, 6, 7\}$ and $V_3 : \Gamma_{\Lambda_3} \to \Gamma_{\Lambda_3}$ we have

$$\begin{aligned}
(V_3\mathbf{x})_1 &= x_1[1 + x_2 - x_3 + x_4 - x_5 + x_6 - x_7] \\
(V_3\mathbf{x})_2 &= x_2[1 - x_1 + x_3 - x_4 + x_5 - x_6 + x_7] \\
(V_3\mathbf{x})_3 &= x_3[1 + x_1 - x_2 + x_4 - x_5 + x_6 - x_7] \\
(V_3\mathbf{x})_4 &= x_4[1 - x_1 + x_2 - x_3 + x_5 - x_6 + x_7] \\
(V_3\mathbf{x})_5 &= x_5[1 + x_1 - x_2 + x_3 - x_4 + x_6 - x_7] \\
(V_3\mathbf{x})_6 &= x_6[1 - x_1 + x_2 - x_3 + x_4 - x_5 + x_7] \\
(V\mathbf{x})_7 &= x_7[1 + x_1 - x_2 + x_3 - x_4 + x_5 - x_6]
\end{aligned} \tag{15}$$

or

$$\begin{aligned}
(V_3\mathbf{x})_1 &= x_1^2 + 2x_1x_2 + 2x_1x_4 + 2x_1x_6 \\
(V_3\mathbf{x})_2 &= x_2^2 + 2x_2x_3 + 2x_2x_5 + 2x_2x_7 \\
(V_3\mathbf{x})_3 &= x_3^2 + 2x_3x_1 + 2x_3x_4 + 2x_3x_6 \\
(V_3\mathbf{x})_4 &= x_4^2 + 2x_4x_2 + 2x_4x_5 + 2x_4x_7 \\
(V_3\mathbf{x})_5 &= x_5^2 + 2x_5x_1 + 2x_5x_3 + 2x_5x_6 \\
(V_3\mathbf{x})_6 &= x_6^2 + 2x_6x_2 + 2x_6x_4 + 2x_6x_7 \\
(V\mathbf{x})_7 &= x_7^2 + 2x_7x_1 + 2x_7x_3 + 2x_7x_5
\end{aligned} \tag{16}$$

Recall that

Definition 5.1. A Volterra QSO V with finite state space X, where $|X| = m$, is called uniform if in any row and respectively in any column of a skew-symmetric matrix $A = (a_{ij})_{i,j=1}^m$ the number of positive entries is equal to the number of negative ones.

It is evident that they are uniform Volterra QSOs if and only if m is odd positive integer.

The QSO (9) considered by Zakharevich is a uniform Volterra operator with $m = 3$.. Thus the QSO (10) is approximated by the sequence of uniform Volterra QSOs V_n defined on finite-dimensional simplexes. We investigate the trajectory behaviour of this operator (10) and prove that it is non-regular transformation.

Let $\omega(\mathbf{x}^{(0)}$ is the set of limit points of trajectory $\{\mathbf{x}^{(n)} : n = 0, 1, \cdots\}$. Below we describe the limit points of arbitrary trajectory. Note that it is easy to see that for any finite-dimensional face Γ_α with $|\alpha| = 2k + 1$, there exist the fixed point $\mathbf{x}_\alpha \in int\Gamma_\alpha$ with $\mathbf{x}_\alpha(i) = \frac{1}{2k+1}$ for all $i \in \alpha$.

Respectively, for any face Γ_α with $|\alpha| = 2k$ the fixed point does not exist in $int\Gamma_\alpha$.

Proposition 5.2. For considered QSO (10) fixed point does not exist in $intS^{Z_+}$.

Proof. For $n = 1$ we have

$$\begin{aligned} (V\mathbf{x})_1 &= x_1[1 + \textstyle\sum_{i=1}^\infty x_{2i} - \sum_{i=1}^\infty x_{2i-1} + x_1] \\ (V\mathbf{x})_2 &= x_2[1 - \textstyle\sum_{i=1}^\infty x_{2i} + \sum_{i=1}^\infty x_{2i-1} - 2x_1 + x_2] \end{aligned} \tag{17}$$

and for $n > 1$ we have

$$\begin{aligned} (V\mathbf{x})_{2n-1} &= x_{2n-1}[1 + \textstyle\sum_{i=1}^{n-1} x_{2i-1} - \sum_{i=1}^{n-1} x_{2i} + \sum_{i=n}^\infty x_{2i} - \sum_{i=n+1}^\infty x_{2i-1}] \\ (V\mathbf{x})_{2n} &= x_{2n}[1 - \textstyle\sum_{i=1}^{n} x_{2i-1} + \sum_{i=1}^{n-1} x_{2i} - \sum_{i=n+1}^\infty x_{2i} + \sum_{i=n+1}^\infty x_{2i-1}] \end{aligned} \tag{18}$$

Let $O = \sum_{i=1}^\infty x_{2i-1}$, and $E = \sum_{i=1}^\infty x_{2i}$ with $O + E = 1$.
Then for $n = 1$ we have

$$\begin{aligned} (V\mathbf{x})_1 &= x_1[1 + E - O + x_1] \\ (V\mathbf{x})_2 &= x_2[1 - E + O + x_2 - 2x_1] \end{aligned} \tag{19}$$

and for $n > 1$ we have

$$\begin{aligned} (V\mathbf{x})_{2n-1} &= x_{2n-1}[1 + E - O + x_{2n-1} + 2\textstyle\sum_{i=1}^{n-1} x_{2i-1} - 2\sum_{i=1}^{n-1} x_{2i}] \\ (V\mathbf{x})_{2n} &= x_{2n}[1 - E + O + x_{2n} - 2\textstyle\sum_{i=1}^{n} x_{2i-1} + 2\sum_{i=1}^{n-1} x_{2i}] \end{aligned} \tag{20}$$

Assume $\bar{\mathbf{x}} \in intS^{\mathbb{N}}$ is an fixed point, i.e. $V(\bar{\mathbf{x}}) = \bar{\mathbf{x}}$. Let $\bar{O} = \sum_{i=1}^\infty \bar{x}_{2i+1}$, and $\bar{E} = \sum_{i=0}^\infty \bar{x}_{2i}$ where $\bar{O} + \bar{E} = 1$.

From (15) for $n = 1$ we have $\bar{x}_1 = \bar{O} - \bar{E}$ and $\bar{x}_2 = \bar{O} - \bar{E}$. From (16) for $n = 2$ we have $\bar{x}_3 = \bar{O} - \bar{E}$ and $\bar{x}_4 = \bar{O} - \bar{E}$. By induction one can prove that for any $n \in \mathbb{N}$ $\bar{x}_n = \bar{O} - \bar{E}$. Thus if $\bar{O} - \bar{E} = 0$, then $\bar{x}_i = 0$ for all i. If $\bar{O} - \bar{E} > 0$, then $\sum_{i=0}^\infty \bar{x}_i = \infty$. Therefore there does not exist fixed point belonging $intS^{\mathbb{N}}$.

Similarly one can prove that there does not exist a fixed point belonging interior of infinite-dimensional face of $S^{\mathbb{N}}$. □

Thus we have proved the following statement.

Theorem 5.3. *Any fixed point belongs to interior of finite dimensional face Γ_α with $|\alpha|$ is odd positive integer, i.e. there does not exist a fixed point belonging to the interior of infinite-dimensional face of $S^{\mathbb{N}}$.*

6. Lyapunov function

A continuous function $\varphi : intS^{m-1} \to R$ is called a Lyapunov function for a QSO V if $\varphi(V(x)) \geq \varphi(x)$ for all x (or $\varphi(V(x)) \leq \varphi(x)$ for all x). A Lyapunov function is a very helpful to describe and upper estimate $\omega(x^0)$. However there is no general recipe on how to find such Lyapunov function. In Ganikhodjaev and Zanin (2004), Ganikhodjaev (1993) and Zakharevich (1978), the authors presented examples of Lyapunov functions for dynamical systems.

Below we will use the following theorem.

Theorem 6.1. *(Ganikhodjaev 1993) Let $\boldsymbol{p} = (p_1, \cdots, p_m)$ be the repelling fixed point of the Volterra QSO. Then there exists a global decreasing Lyapunov function of the form $\varphi_p(\boldsymbol{x}) = x_1^{p_1} \cdots x_m^{p_m}$, where $p_i \geq 0,\ \sum_{i=1}^{m} p_i = 1,\ \boldsymbol{p} \in S^{m-1}$.*

Let us consider QSO $V_{2n+1} : \Gamma_{\Lambda_{2n+1}} \to \Gamma_{\Lambda_{2n+1}}$.
As shown above a point $M_n = (\frac{1}{2n+1}, \frac{1}{2n+1}, \cdots, \frac{1}{2n+1}) \in int\Gamma_{\Lambda_{2n+1}}$ is a fixed point. Let us show that this point is the repelling fixed point for arbitrary n. The Jacobian $J_V(M_n)$ is defined as follows:

$$J_V(M_n) = \begin{pmatrix} 1 & \frac{1}{2n+1} & -\frac{1}{2n+1} & \cdots & \frac{1}{2n+1} & -\frac{1}{2n+1} \\ -\frac{1}{2n+1} & 1 & \frac{1}{2n+1} & \cdots & -\frac{1}{2n+1} & \frac{1}{2n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ -\frac{1}{2n+1} & \frac{1}{2n+1} & -\frac{1}{2n+1} & \cdots & 1 & \frac{1}{2n+1} \\ \frac{1}{2n+1} & -\frac{1}{2n+1} & \frac{1}{2n+1} & \cdots & -\frac{1}{2n+1} & 1 \end{pmatrix}. \tag{21}$$

It is known that for Volterra quadratic stochastic operator (6) defined on finite-dimensional simplex one of the eigenvalues of corresponding Jacobian is equal to 1 (see Devaney (2003) and Ganikhodjaev *et al.* (2015)).

Theorem 6.2. *One of the eigenvalues of $J_V(M_n)$ is equal to 1 and for all other eigenvalues we have $|\lambda| > 1$ that is the fixed point M_n is repelling.*

Proof. Let us consider a matrix

$$\begin{pmatrix} 1-\lambda & b & -b & \cdots & b & -b \\ -b & 1-\lambda & b & \cdots & -b & b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ -b & b & -b & \cdots & 1-\lambda & b \\ b & -b & b & \cdots & -b & 1-\lambda \end{pmatrix} \tag{22}$$

where $b = \frac{1}{2n+1}$.
For $n = 1$ we have the characteristic equation $(1-\lambda)^3 + \frac{1}{3}(1-\lambda) = 0$, with $\lambda_1 = 1$ and $(1-\lambda)^2 = -\frac{1}{3}$, i.e. $|\lambda_{2,3}| > 1$.
For $n = 2$ we have the characteristic equation $(1-\lambda)^5 + \frac{2}{5}(1-\lambda)^3 + \frac{1}{125}(1-\lambda) = 0$, with $\lambda_1 = 1$, and $(1-\lambda)^2 < 0$, i.e. $|\lambda_{2,3,4,5}| > 1$.
For $n = 3$ we have the characteristic equation $(1-\lambda)^7 + \frac{3}{7}(1-\lambda)^5 + \frac{5}{343}(1-\lambda)^3 + \frac{1}{16807}(1-\lambda) = 0$, with $\lambda_1 = 1$ and $(1-\lambda)^2 < 0$, i.e. $|\lambda_{2,3,4,5,6,7}| > 1$.
For arbitrary n we have the characteristic equation

$$(1-\lambda)^{2n+1} + a_1(1-\lambda)^{2n-1} + a_2(1-\lambda)^{2n-3} + \cdots + a_{n-1}(1-\lambda)^3 + a_n(1-\lambda) = 0,$$

where all coefficients $a_i > 0, i = 1, \cdots, n$. Thus $\lambda_1 = 1$ and since $(1-\lambda)^2 < 0$, we have $|\lambda_i| > 1$ for $i = 2, 3, \cdots, 2n+1$. □

According to Theorem (6.1) the following function $\varphi_{M_n}^{(n)}(\mathbf{x}_n) = (x_1 \cdot x_2 \cdots x_{2n+1})^{\frac{1}{2n+1}}$ is the decreasing Lyapunov function for arbitrary $\mathbf{x}_n \in int\Gamma_{\Lambda_{2n+1}}$.

Now for arbitrary $\mathbf{x} \in S^{\mathbb{N}}$ let $\mathbf{x}_n = (x_1, \cdots, x_{2n}, 1 - \sum_{i=1}^{2n} x_i) \in \Gamma_{\Lambda_{2n+1}}$ and $\varphi_{M_n}^{(n)}(\mathbf{x}_n) = (x_1 \cdot x_2 \cdots x_{2n+1})^{\frac{1}{2n+1}}$ is the decreasing Lyapunov function for $\mathbf{x}_n \in int\Gamma_{\Lambda_{2n+1}}$, i.e. $\varphi_{M_n}^{(n)}(V_{2n+1}\mathbf{x}_n) < \varphi_{M_n}^{(n)}(\mathbf{x}_n)$. It is follow from well-known AM-GM inequality. For example, for $n = 1$,

$$(x_1[1 + x_2 - x_3] \cdot x_2[1 - x_1 + x_3] \cdot x_3[1 + x_1 - x_2])^{\frac{1}{3}} < (x_1 x_2 x_3)^{\frac{1}{3}} \,. \tag{23}$$

Thus $\varphi_{M_n}^{(n)}(V_{2n+1}^k \mathbf{x}_n) \to 0$ for $k \to \infty$.

Finally we define a global Lyapunov function on $intS^{\mathbb{N}}$ as the following sum

$$\varphi(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{1}{2^{2n+1}} \varphi_{M_n}^{(n)}(\mathbf{x}_n).$$

Then $\varphi(V^k\mathbf{x}) \to 0$ for $k \to \infty$, that is, the limit set lies on $\partial S^{\mathbb{N}}$. Since Volterra operators leave the vertices of $\partial S^{\mathbb{N}}$ fixed and the faces invariant, the limit set cannot be finite.

Thus we have proved the following statement.

Theorem 6.3. *The QSO V (10) is non-regular transformation.*

7. Conclusion

We present Volterra quadratic stochastic operator defined on infinite state space $\mathbb{N} = \{1, 2, \cdots\}$ of positive integers and prove that it is non-regular transformation. One can expect that it is also a non-ergodic transformation. Discussion of this rather difficult problem will be the subject of future article.

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