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SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS USING FRACTIONAL EXPLICIT METHOD

(Menyelesaikan Persamaan Pembezaan Pecahan Menggunakan Kaedah Tak Tersirat Pecahan)

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ABSTRACT

This research is focusing in solving the fractional differential equations (FDEs) for linear and non-linear type using fractional explicit method (FEM) with constant step-size. Most of the numerical methods for solving FDEs involved the interpolating points of step size h. Some modifications were implemented in the derivation technique, where the step size $2h$ are considered in the formula of the proposed method. The main goal of this research is to derive FEM by considering the implementation of second-order Adam-Bashforth method using Lagrange interpolation for fractional case. Besides, the order and convergence analysis of the developed method will also be investigated in this study. The algorithm of the proposed method is written in C language. Based on the numerical results obtained, it is clearly ratified that the proposed method converges as the step size, h is getting smaller in solving the FDEs.

Keywords: fractional differential equations; linear FDE; nonlinear FDE; fractional Riccati differential equation; single order FDE

ABSTRAK

Kajian ini memberi tumpuan dalam menyelesaikan persamaan pembezaan pecahan (PPP) bagi jenis linear dan bukan linear menggunakan kaedah tak tersirat pecahan (KTTP) untuk langkah malar. Kebanyakkan kaedah berangka bagi penyelesaian PPP melibatkan titik interpolasi saiz langkah h. Beberapa pengubahsuaian telah dilaksanakan dalam teknik derivasi, di mana saiz langkah $2h$ dipertimbangkan dalam formula kaedah yang dicadangkan. Matlamat utama kajian ini adalah untuk menerbitkan KTTP dengan mempertimbangkan pelaksanaan kaedah Adam-Bashforth peringkat kedua menggunakan interpolasi Lagrange untuk kes pecahan. Selain itu, peringkat dan analisis konvergen bagi kaedah yang dibangunkan juga akan disiasat dalam kajian ini. Algoritma bagi kaedah yang dibangunkan ditulis dalam bahasa C. Berdasarkan hasil berangka yang diperolehi, jelas menunjukkan bahawa kaedah yang dicadangkan menumpu apabila saiz langkah h semakin kecil dalam menyelesaikan PPP.

Kata kunci: persamaan perbezaan pecahan; linear PPP; bukan linear PPP; persamaan perbezaan pecahan Riccati; PPP peringkat satu

1. Introduction

The FDEs are crucial in a wide variety of disciplines, including financial economics, modelling of materials and diffusion processes. Besides, Troparevsky *et al*. (2019) claimed that FDEs are useful in modelling cases in the field of science and engineering as it is able in capturing nonlocality properties. This is because FDEs are not only consider the local aspects of the dynamics but also the global development of the system. As a result, they will give more accurate approximations of real-world behaviour when compared to standard derivatives. Therefore, the solutions for FDEs have received substantial attention due to their importance in various fields.

According to Biala and Jator (2015b), the equation of FDE is in the form of:

$$ {}_{C}D_{s_0}^{\alpha} y(s) = f(s, y(s)), \qquad y(s_0) = y_0. \tag{1}$$

with the order is $0 < \alpha < 1$.

In fractional calculus, there are multiple kinds of fractional differential operators, such as the Riemann-Liouville differential operator, the Grünwald-Letnikov differential operator and the Caputo differential operator. However, the most common differential operators used in FDEs are the Riemann-Liouville differential operator and Caputo differential operator. Garappa (2009) defined ${}_{C}D_{t_0}^{\alpha}$ as the fractional Caputo's α-derivative operator:

$$ {}_{C}D_{s_0}^{\alpha} = {}_{RL}D_{s_0}^{\alpha} y(s) - y(s_0), \tag{2}$$

with ${}_{RL}D_{s_0}^{\alpha} y(s)$ is the Riemann-Liouville differential operator as:

$$ {}_{RL}D_{s_0}^{\alpha} y(s) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{ds}\right)^{m} \int_{s_0}^{s} \frac{y(\tau)d\tau}{(s-\tau)^{\alpha-m+1}}, \qquad \alpha > 0, m = \lceil \alpha \rceil. \tag{3}$$

Many of the FDEs cannot be solved analytically, which means that FDEs do not have exact solutions. There are situations in which it has been shown that solving FDEs numerically is a more effective and convenient method compared to solve them analytically. This is particularly when solving the problems which are very complex and enormous. Thus, numerical methods have become more crucial when it comes to finding the solutions for the FDEs. In literature, some researchers have developed few numerical methods to solve FDEs in the past few years. Biala and Jator (2015b) has proposed a family of Implicit Adams Methods (IAMS) for solving FDE and the results shown that the errors decrease when α increases. Besides, Ahmed (2018) proposed the modified fractional Euler method (MFEM) and the outcome shown that when the step size decreases, the accuracy improved. Then, Bonab and Javidi (2020) discussed a family of multistep explicit method for solving FDEs. They managed to show that the interval of stability has been enhanced when the larger stability region is used. Zabidi *et al.* (2022) has proposed an Adams-type multistep method in predict-correct technique for solving differential equations of fractional order.

The goal of this study is to derive FEM in order to approximate the solutions for linear and nonlinear FDEs. Most of the existing numerical methods were derived involving the interpolating points of step size h. In this study, we aim to implement some modifications in the derivation technique by considering the step size of $2h$ in the formula of the proposed method. The derivation is based on Adams-Bashforth method of fractional case by which the type of differential operator used in this derivation is the Caputo differential operator. Next, the derivation also implements Lagrange interpolation for fractional case.

2. Fractional Explicit Method (FEM)

The derivation of FEM will be described in this section by first taking into consideration the fractional initial value problems (FIVP) as in the form:

$$ D^{\alpha} y(s) = f\big(s, y(s)\big), \quad y^{k}(0) = y_0^{k}, \qquad k = 0,1, \dots, \lceil \alpha \rceil - 1. \tag{4}$$

Eq. (4) can be rewritten equivalent to the Volterra integral equation as follows,

$$y(s) = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{s^k}{k!} y^k(0) + \frac{1}{\Gamma(\alpha)} \int_0^s [(s-\tau)^{\alpha-1} f(\tau, y(\tau))]\, d\tau. \tag{5}$$

According to Diethelm (2010), the methods for FDEs can be constructed by considering the methods for classical first-order equations and generalizing the concept in an approximate way. Thus, we get Eq. (4) as:

$$Dy(s) = f(s, y(s)), \quad y(0) = y_0. \tag{6}$$

Simplifying Eq. (5) will yield:

$$\mathrm{y(s)} = \mathrm{y}_0 + \frac{1}{\Gamma(\alpha)} \int_0^s [(\mathrm{s}-\tau)^{\alpha-1} \mathrm{f}(\tau, \mathrm{y}(\tau))]\, \mathrm{d}\tau. \tag{7}$$

Then, at the points $s = s_{n+1}$ and $s = s_n$, the approximation solutions have been proposed. Therefore,

(1) If $s = s_{n+1}$,

$$y(s_{n+1}) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{s_{n+1}} [(s_{n+1}-\tau)^{\alpha-1} f(\tau, y(\tau))]\, d\tau. \tag{8}$$

(2) If $s = s_n$,

$$y(s_n) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{s_n} [(s_n-\tau)^{\alpha-1} f(\tau, y(\tau))]\, d\tau. \tag{9}$$

Subtracting Eq. (8) with Eq. (9) will get:

$$y(s_{n+1}) = y(s_n) + \frac{1}{\Gamma(\alpha)} \left[\int_0^{s_{n+1}} [(s_{n+1}-\tau)^{\alpha-1} f(\tau, y(\tau))]\, d\tau - \int_0^{s_n} [(s_n-\tau)^{\alpha-1} f(\tau, y(\tau))]\, d\tau \right]. \tag{10}$$

The proposed method of FEM is of order 2. In order to evaluate the approximation solutions, two interpolating functions which are F_n and F_{n-2} are considered for Lagrange interpolation. Then, we have:

$$P(s) \left(\approx f\big(\tau, y(\tau)\big) \right) = \frac{s - s_{n-2}}{s_n - s_{n-2}} F_n + \frac{s - s_n}{s_{n-2} - s_n} F_{n-2}. \tag{11}$$

Next, we let:

$$2h = s_n - s_{n-2}, \qquad \tau = s. \tag{12}$$

Eqs. (11) - (12) will be substituted into Eq. (10), yielding:

$$
\begin{aligned}
y(s_{n+1}) &= y(s_n) \\
&+\frac{1}{\Gamma(\alpha)}\left[\int_0^{s_{n+1}}(s_{n+1}-s)^{\alpha-1}\left(\frac{s-s_{n-2}}{s_n-s_{n-2}}F_n+\frac{s-s_n}{s_{n-2}-s_n}F_{n-2}\right)ds\right] \\
&-\frac{1}{\Gamma(\alpha)}\left[\int_0^{s_n}(s_n-s)^{\alpha-1}\left(\frac{s-s_{n-2}}{s_n-s_{n-2}}F_n+\frac{s-s_n}{s_{n-2}-s_n}F_{n-2}\right)ds\right].
\end{aligned}
\tag{13}
$$

Then, evaluating the first fractional integral as:

$$
\begin{aligned}
&\int_0^{s_{n+1}}[(s_{n+1}-s)^{\alpha-1}f(\tau,y(\tau))]\,d\tau \\
&=\sum_{p=0}^{n}\int_{s_p}^{s_{p+1}}(s_{n+1}-s)^{\alpha-1}\left(\frac{s-s_{n-2}}{s_n-s_{n-2}}F_n+\frac{s-s_n}{s_{n-2}-s_n}F_{n-2}\right)\,ds, \\
&=\sum_{p=0}^{n}\left[\frac{F_n}{s_n-s_{n-2}}\int_{s_p}^{s_{p+1}}(s_{n+1}-s)^{\alpha-1}\,(s-s_{n-2})\,ds+\right. \\
&\left.\frac{F_{n-2}}{s_{n-2}-s_n}\int_{s_p}^{s_{p+1}}(s_{n+1}-s)^{\alpha-1}\,(s-s_n)\,ds\right], \\
&=\sum_{p=0}^{n}\left[\frac{F_n}{2h}\int_{s_p}^{s_{p+1}}(s_{n+1}-s)^{\alpha-1}\,(s-s_{n-2})\,ds-\right. \\
&\left.\frac{F_{n-2}}{2h}\int_{s_p}^{s_{p+1}}(s_{n+1}-s)^{\alpha-1}\,(s-s_n)\,ds\right].
\end{aligned}
\tag{14}
$$

Then, we consider the change of variables where $y=s_{n+1}-s, dy=-ds$ and substitute into Eq. (14). We obtain:

$$
\begin{aligned}
&\sum_{p=0}^{n}\left[\frac{F_n}{2h}\int_{s_{n+1}-s_p}^{s_{n+1}-s_{p+1}}(y)^{\alpha-1}\,(s_{n+1}-y-s_{n-2})\,(-dy)-\right. \\
&\qquad\left.\frac{F_{n-2}}{2h}\int_{s_{n+1}-s_p}^{s_{n+1}-s_{p+1}}(y)^{\alpha-1}\,(s_{n+1}-y-s_n)\,(-dy)\right], \\
&=\sum_{p=0}^{n}\left\{\frac{F_n}{2h}\left[-\int_{s_{n+1}-s_p}^{s_{n+1}-s_{p+1}}(y)^{\alpha-1}\,(s_{n+1}-y-s_{n-2})\,dy\right]-\right. \\
&\qquad\left.\frac{F_{n-2}}{2h}\left[-\int_{s_{n+1}-s_p}^{s_{n+1}-s_{p+1}}(y)^{\alpha-1}\,(s_{n+1}-y-s_n)\,dy\right]\right\}, \\
&=\frac{F_n}{2h}\left[-\frac{3h}{\alpha}[(s_{n+1}-s_{n+1})^{\alpha}-(s_{n+1}-s_0)^{\alpha}]+\right. \\
&\qquad\left.\frac{1}{\alpha+1}[(s_{n+1}-s_{n+1})^{\alpha+1}-(s_{n+1}-s_0)^{\alpha+1}]\right]- \\
&\quad\frac{F_{n-2}}{2h}\left[-\frac{h}{\alpha}[(s_{n+1}-s_{n+1})^{\alpha}-(s_{n+1}-s_0)^{\alpha}]+\right.
\end{aligned}
\tag{15}
$$

$$\frac{1}{\alpha+1}\left[(s_{n+1}-s_{n+1})^{\alpha+1}-(s_{n+1}-s_0)^{\alpha+1}\right]\Bigg],$$

$$=\frac{F_n}{2h}\left[h(h^\alpha)\left(\frac{3(n+1)^\alpha}{\alpha}-\frac{(n+1)^{\alpha+1}}{\alpha+1}\right)\right]$$

$$-\frac{F_{n-2}}{2h}\left[h(h^\alpha)\left(\frac{(n+1)^\alpha}{\alpha}-\frac{(n+1)^{\alpha+1}}{\alpha+1}\right)\right],$$

$$=h^\alpha\left[\left(\frac{3(n+1)^\alpha}{2\alpha}-\frac{(n+1)^{\alpha+1}}{2(\alpha+1)}\right)F_n-\left(\frac{(n+1)^\alpha}{2\alpha}-\frac{(n+1)^{\alpha+1}}{2(\alpha+1)}\right)F_{n-2}\right]. \qquad (16)$$

Using the above same steps, we evaluate the second fractional integral. Thus, we obtain:

$$\int_0^{s_n}\left[(s_n-s)^{\alpha-1}f(\tau,y(\tau))\right]d\tau$$

$$=\sum_{p=0}^{n-1}\int_{s_p}^{s_{p+1}}(s_n-s)^{\alpha-1}\left(\frac{s-s_{n-2}}{s_n-s_{n-2}}F_n+\frac{s-s_n}{s_{n-2}-s_n}F_{n-2}\right)ds, \qquad (17)$$

$$=\sum_{p=0}^{n-1}\left[\frac{F_n}{s_n-s_{n-2}}\int_{s_p}^{s_{p+1}}(s_n-s)^{\alpha-1}(s-s_{n-2})\,ds+\frac{F_{n-2}}{s_{n-2}-s_n}\int_{s_p}^{s_{p+1}}(s_n-s)^{\alpha-1}(s-s_n)\,ds\right],$$

$$=\sum_{p=0}^{n-1}\left[\frac{F_n}{2h}\int_{s_p}^{s_{p+1}}(s_n-s)^{\alpha-1}(s-s_{n-2})\,ds-\frac{F_{n-2}}{2h}\int_{s_p}^{s_{p+1}}(s_n-s)^{\alpha-1}(s-s_n)\,ds\right]. \qquad (18)$$

Next, by considering the changes $y=s_n-s, dy=-ds$, we will obtain:

$$\sum_{p=0}^{n-1}\left[\frac{F_n}{2h}\int_{s_n-s_p}^{s_n-s_{p+1}}(y)^{\alpha-1}(s_n-y-s_{n-2})(-dy)-\frac{F_{n-2}}{2h}\int_{s_n-s_p}^{s_n-s_{p+1}}(y)^{\alpha-1}(s_n-y-s_n)(-dy)\right],$$

$$=\sum_{p=0}^{n-1}\left\{\frac{F_n}{2h}\left[-\int_{s_n-s_p}^{s_n-s_{p+1}}(y)^{\alpha-1}(s_n-y-s_{n-2})\,dy\right]-\frac{F_{n-2}}{2h}\left[-\int_{s_n-s_p}^{s_n-s_{p+1}}(y)^{\alpha-1}(s_n-y-s_n)\,dy\right]\right\},$$

$$=\frac{F_n}{2h}\Bigg[-\frac{2h}{\alpha}\left[(s_n-s_n)^\alpha-(s_n-s_0)^\alpha\right]+$$

$$\frac{1}{\alpha+1}\left[(s_n-s_n)^{\alpha+1}-(s_n-s_0)^{\alpha+1}\right]\Bigg]-$$

$$\frac{F_{n-2}}{2h}\left[\frac{1}{\alpha+1}[(s_n-s_n)^{\alpha+1}-(s_n-s_0)^{\alpha+1}]\right],$$
$$=\frac{F_n}{2h}\left[h(h^\alpha)\left(\frac{2(n)^\alpha}{\alpha}-\frac{(n)^{\alpha+1}}{\alpha+1}\right)\right]+\frac{F_{n-2}}{2h}\left[h(h^\alpha)\left(\frac{(n)^{\alpha+1}}{\alpha+1}\right)\right],$$
$$=h^\alpha\left[\left(\frac{2(n)^\alpha}{2\alpha}-\frac{(n)^{\alpha+1}}{2(\alpha+1)}\right)F_n+\left(\frac{(n)^{\alpha+1}}{2(\alpha+1)}\right)F_{n-2}\right]. \tag{19}$$

The numerical formula for FEM can be obtained when Eq. (16) and Eq. (19) will be substituted into Eq. (10). Therefore, we have:

$$y(s_{n+1})=y(s_n)+\frac{h^\alpha}{\Gamma(\alpha)}\left\{\left[\left(\frac{3(n+1)^\alpha}{2\alpha}-\frac{(n+1)^{\alpha+1}}{2(\alpha+1)}\right)F_n-\left(\frac{(n+1)^\alpha}{2\alpha}-\frac{(n+1)^{\alpha+1}}{2(\alpha+1)}\right)F_{n-2}\right]-\left[\left(\frac{2(n)^\alpha}{2\alpha}-\frac{(n)^{\alpha+1}}{2(\alpha+1)}\right)F_n+\left(\frac{(n)^{\alpha+1}}{2(\alpha+1)}\right)F_{n-2}\right]\right\},$$

$$y(s_{n+1})=y(s_n)+\frac{h^\alpha}{\Gamma(\alpha)}\left[\left(\frac{3(n+1)^\alpha-2(n)^\alpha}{2\alpha}+\frac{(n)^{\alpha+1}-(n+1)^{\alpha+1}}{2(\alpha+1)}\right)F_n+\left(\frac{-(n+1)^\alpha}{2\alpha}+\frac{(n+1)^{\alpha+1}-(n)^{\alpha+1}}{2(\alpha+1)}\right)F_{n-2}\right]. \tag{20}$$

Thus, the developed numerical scheme known as FEM is shown in Eq. (20).

3. Analysis of the Method

3.1. *Order of the developed method*

Definition 1. (Galeone & Garrappa 2006). The fractional linear multistep method (FLMM) can be written in the form of:

$$\sum_{j=0}^{n}\alpha_j y_{n-j}=h^\alpha\sum_{j=0}^{n}\beta_j f(s_{n-j},y_{n-j}). \tag{21}$$

where α_j and β_j are coefficients of real parameters and α denotes the fractional order.

Definition 2. (Lambert 1973). The order of the developed method has order w, if $C_0=C_1=\cdots=C_w=0$ and $C_{w+1}\neq 0$ is the error constant. The formula is given as:

$$C_w=\sum_{j=0}^{k}\left[\frac{j^w\alpha_j}{w!}-\frac{j^{w-1}\beta_j}{(w-1)!}\right],\quad w=0,1,2,\dots \tag{22}$$

where α and β are the coefficients from the developed method, and k is the order of the developed method.

The first step to calculate the order for FEM in Eq. (20) is to determine α_j and β_j using Eq. (20) and Eq. (21). Thus, we get:

$$\begin{aligned} &\alpha_0 = 0, \quad \beta_0 = \frac{1}{\Gamma(\alpha)}\left(\frac{-(n+1)^{\alpha}}{2\alpha} + \frac{(n+1)^{\alpha+1} - (n)^{\alpha+1}}{2(\alpha+1)}\right), \\ &\alpha_1 = 0, \quad \beta_1 = 0, \\ &\alpha_2 = -1, \quad \beta_2 = \frac{1}{\Gamma(\alpha)}\left(\frac{3(n+1)^{\alpha} - 2(n)^{\alpha}}{2\alpha} + \frac{(n)^{\alpha+1} - (n+1)^{\alpha+1}}{2(\alpha+1)}\right), \\ &\alpha_3 = 0, \quad \beta_3 = 0. \end{aligned} \tag{23}$$

Next, substitute Eq. (23) into Eq. (22) will yield:

$$\begin{aligned} C_0 &= \sum_{j=0}^{k} \alpha_j = 0, \\ C_1 &= \sum_{j=0}^{k} \left(j\alpha_j - \beta_j\right) = 0, \\ C_2 &= \sum_{j=0}^{k} \left(\frac{j^2\alpha_j}{2!} - j\beta_j\right) = 0, \\ C_3 &= \sum_{j=0}^{k} \left(\frac{j^3\alpha_j}{3!} - \frac{j^2\alpha_j}{2!}\right) = \frac{2}{3}. \end{aligned} \tag{24}$$

Thus, the developed method which is FEM is of order 2 and the error constant is $\frac{2}{3}$.

3.2. *Convergence analysis*

Theorem 1. *Let $\gamma_1, \gamma_2, \dots, \gamma_p$ are the roots for the characteristic's equation,*

$$a \le t \le b, -\infty < y < \infty, \tag{25}$$

such that a and b are finite. Suppose that there exists a constant L such that, for every s, y, y^*, the coordinates s, y, y^* and (s, y^*) are both in R where,

$$|f(s,y) - f(s,y^*)| \le L|y - y^*|. \tag{26}$$

Theorem 2. (Biala & Jator, 2015a; Diethelm, 2010; Li & Tao, 2009). *A linear multistep method is said to be convergent if, for all initial values problems subject to the hypothesis of Theorem 1 as $s \in [a,b]$ and $0 < \alpha < 1$, we have that,*

$$|y - y^*| \le K \cdot s^{\alpha-1} h^p, \tag{27}$$

where K is a constant depending only on α and p as $p \in (0,1)$ and,

$$\lim_{h\to 0} y_n = y^*(s_n). \tag{28}$$

The first step is to recall the proposed method based on Eq. (20) and let:

$$P = \frac{3(n+1)^{\alpha} - 2(n)^{\alpha}}{2\alpha} + \frac{(n)^{\alpha+1} - (n+1)^{\alpha+1}}{2(\alpha+1)},$$
$$Q = \frac{-(n+1)^{\alpha}}{2\alpha} + \frac{(n+1)^{\alpha+1} - (n)^{\alpha+1}}{2(\alpha+1)}. \tag{29}$$

Next, substitute Eq. (29) into Eq. (20) will yield:

(1) The exact form of the system is given by:

$$y^*(s_{n+1}) - y^*(s_n) = \frac{h^{\alpha}}{\Gamma(\alpha)}(P)F_n^* + \frac{h^{\alpha}}{\Gamma(\alpha)}(Q)F_{n-2}^* + \frac{2}{3}h^3 s^{*(3)}(\varepsilon). \tag{30}$$

(2) The approximate form of the system is given by:

$$y(s_{n+1}) - y(s_n) = \frac{h^{\alpha}}{\Gamma(\alpha)}(P)F_n + \frac{h^{\alpha}}{\Gamma(\alpha)}(Q)F_{n-2}. \tag{31}$$

Subtracting Eq. (31) and Eq. (30) will get:

$$\begin{aligned} y(s_{n+1}) - y^*(s_{n+1}) &= y(s_n) - y^*(s_n) + \frac{h^{\alpha}}{\Gamma(\alpha)}(P)(F_n - F_n^*) \\ &\quad + \frac{h^{\alpha}}{\Gamma(\alpha)}(Q)(F_{n-2} - F_{n-2}^*) + \frac{2}{3}h^3 y^{*(3)}(\varepsilon), \\ y(s_{n+1}) - y^*(s_{n+1}) &= y(s_n) - y^*(s_n) + \frac{h^{\alpha}}{\Gamma(\alpha)}(P)[f(s_n, y_n) - f(s_n^*, y_n^*)] \\ &\quad + \frac{h^{\alpha}}{\Gamma(\alpha)}(Q)[f(s_{n-2}, y_{n-2}) - f(s_{n-2}^*, y_{n-2}^*)] \\ &\quad + \frac{2}{3}h^3 y^{*(3)}(\varepsilon). \end{aligned} \tag{32}$$

Let:

$$\begin{aligned} |d_{n+1}| &= |y_{n+1} - y_{n+1}^*|, \\ |d_n| &= |y_n - y_n^*|, \\ |d_{n-2}| &= |y_{n-2} - y_{n-2}^*|. \end{aligned} \tag{33}$$

Then, we use the assumption in Eq. (32) and Theorem 1 above by applying the Lipschitz condition. Thus, we have:

$$|d_{n+1}| \le \left(1 + \frac{h^{\alpha}P}{\Gamma(\alpha)}\right)|d_n| + \frac{h^{\alpha}Q}{\Gamma(\alpha)}|d_{n-2}| + \frac{2}{3}h^3 y^{*(3)}(\varepsilon). \tag{34}$$

Rewriting Eq. (34) based on Theorem 2 above will yield:

$$|d_{n+1}| \le (1 + Kh^{\alpha})|d_n| + Kh^{\alpha}|d_{n-2}| + \frac{2}{3}h^3 y^{*(3)}(\varepsilon). \tag{35}$$

From the above analysis, we observe that as h is sufficiently small or $h \to 0$ and the initial value tends to 0, it is proven that $|d_{n+1}| \leq |d_n|$; thus, we will get $|y_{n+1}| = |y^*_{n+1}|$ and $|y_n| = |y^*_n|$. In a nutshell, Theorem 4 is satisfied and hence, the proposed method, FEM, is proved to converge.

3.3. *Stability of the method*

Definition 3. Let $\gamma_1,\ \gamma_2, \ldots, \gamma_p$ are the roots for the characteristic's equation,

$$P(\gamma) = \gamma^p - a_{p-1}\gamma^{p-1} - \cdots - a_{p-1}\gamma - a_0, \tag{36}$$

for the given p-step multistep method,

$$\begin{aligned} y_{n+1} = a_{p-1}y_n + a_{p-2}y_{n-1} + \cdots + a_0 y_{n+1-p} \\ + h\big[b_p f(s_{n+1}, y_{n+1}) + b_{p-1} f(s_n, y_n) + \cdots \\ + b_0 f(s_{n+1-p}, y_{n+1-p)}\big]. \end{aligned} \tag{37}$$

If all the roots have value 1 are simple roots, then the root condition of the difference equation is said to be satisfied. Moreover, the methods are strongly stable if the methods have the only root $\gamma = 1$ and satisfy the root condition.

The first step is recalling the developed method in Eq. (20). Then, the equation for the general 2-step multistep method based on Eq. (37) is:

$$\begin{aligned} y_{n+1} = a_1 y_n + a_0 y_{n-1} \\ + h[b_2 f(s_{n+1}, y_{n+1}) + b_1 f(s_n, y_n) + b_0 f(s_{n-1}, y_{n-1})]. \end{aligned} \tag{38}$$

Next, the general characteristics equation for $m = 2$ based on Eq. (36) is:

$$P(\gamma) = \gamma^2 - a_1\gamma^1 - a_{0.} \tag{39}$$

Then, comparing Eq. (20) and Eq. (39), we get:

$$a_1 = 1, a_0 = 0. \tag{40}$$

Substituting Eq. (40) into Eq. (39), we get the characteristics polynomial of the developed method as:

$$\begin{aligned} &P(\gamma) = \gamma^2 - \gamma = 0, \\ &\gamma\ (\gamma - 1) = 0, \\ &\gamma = 0, \qquad \gamma = 1. \end{aligned} \tag{41}$$

The proposed method is said to be strongly stable as the above characteristic's polynomial satisfies the root condition.

4. Implementation

4.1. *Algorithm of the method*

The algorithm for the developed method will be included in this section. The algorithm will be written in C language. First and foremost, the inputs for this programming are the values of lower limits and upper limits, the step size and the value of α. On the other hand, the output values are the values for the approximation of y. The following is the algorithm for the proposed method.

Step 1: Set lower limit, upper limit, α = alpha, $\Gamma(\alpha)$ = gamma and the number of iterations, $w = \frac{upper\ limit - lower\ limit}{h}$.

Step 2: For $n = 0$, the approximation solution of y_1 is calculated using Fractional Euler method: $y(s_{n+1}) = y(s_n) + \frac{h^\alpha}{\Gamma(\alpha+1)}(F_n)$.

Step 3: For $n = 1, 2, 3, \ldots, w-1$, calculate the approximation value of $y_2, y_3, \ldots, y_{w-1}$ by iterating the procedure of steps 4 – 6 until y_{w-1} is achieved.

Step 4: Set $s_{n+1} = s_n + h$.

Step 5: Next, find the approximation value of y_{n+1} by using the proposed method.

Step 6: Find the absolute error.
Note that $error = |y_{n+1} - Y_{n+1}|$ where y_{n+1} is the approximation solution and Y_{n+1} is the exact solution.

Step 7: End.

5. Numerical Examples and Discussion

The performance of the developed method will be validated by four numerical examples which consist of different types of FDE problem. Problem 1 and Problem 2 are nonlinear FDE, which have an exact solution when $\alpha = 1.0$. Problem 3 is a fractional Riccati differential equations while Example 4 is FDE with variable order of α.

Problem 1. Consider a problem of nonlinear FDE (Al-Rabtah *et al.* 2012).

$$D^\alpha y(s) = (1 - y(s))^4, \qquad y(0) = 0.$$

Exact solution is $y(s) = \frac{1+3s-\left(1+6s+9s^2\right)^{\frac{1}{3}}}{(1+3s)}$ as $\alpha = 1.0$.

Problem 2. Consider a problem of nonlinear FDE (Lydia *et al.* 2021).

$$D^\alpha y(s) = y^2(s) + 1, \qquad y(0) = 0.$$

Exact solution is $y(s) = \tan(s)$ for $\alpha = 1.0$.

Problem 3. Consider the fractional Riccati differential equations as an application problem (Odibat & Momani 2008; Merden 2012).

$$D^\alpha y(s) = -y^2(s) + 1, \qquad y(0) = 0.$$

Exact solution is $y(s) = \frac{e^{2s}-1}{e^{2s}+1}$ for $\alpha = 1.0$.

Problem 4. Consider a problem of nonlinear FDE (Bonab & Javidi 2020).

$$D^{\alpha}y(s) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}s^{\alpha} - \frac{2}{\Gamma(3-\alpha)}s^{2-\alpha} + (s^{2\alpha} - s^2)^4 - y^4(s),$$
$$y(0) = 0.$$

Exact solution is $y(s) = s^{2\alpha} - s^2$.

The notations used in the tables are listed below.

h	Step size
FEM	Fractional explicit method of order 2 proposed in this research
TSLAB	Two-step Laplace Adam-Bashforth method order 2 (Gnitchogna & Atangana 2017)
MAXE	Maximum error
AVGE	Average error

Tables 1 - 3 represent the numerical solutions by measuring the absolute error at each point when $\alpha = 1.00$ with different step size, h using the developed method FEM and the existing method two-step Laplace Adam-Bashforth (TSLAB) to solve Problems 1 – 3 respectively. The absolute errors for both methods are comparable. Figures 1 - 3 present the behaviour of approximate solutions of FEM at different values of step size, h. As we can observed from the three tables, absolute error is getting smaller as the step size, h decreases. It implies that when the step size, h is smaller, the approximate solutions converged as it approaches to the exact solutions. Besides, as h decreases, the average error and maximum error are getting smaller. Therefore, this indicates that FEM is performing well for solving the nonlinear FDEs compared to the exact solution when $\alpha = 1.00$.

Table 1: Numerical solutions for Problem 1 when $\alpha = 1.0$ at different step sizes, h

S	Exact	Absolute error at $h = 0.1$		Absolute error at $h = 0.01$		Absolute error at $h = 0.001$	
		FEM	TSLAB	FEM	TSLAB	FEM	TSLAB
0.0	0.00000	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}
0.1	0.08374	1.6×10^{-2}	1.6×10^{-2}	1.9×10^{-4}	8.2×10^{-5}	1.8×10^{-6}	7.8×10^{-7}
0.2	0.14501	2.1×10^{-2}	3.4×10^{-3}	9.7×10^{-5}	3.1×10^{-5}	9.0×10^{-7}	2.9×10^{-7}
0.3	0.19261	8.6×10^{-3}	1.9×10^{-3}	4.9×10^{-5}	7.8×10^{-6}	4.5×10^{-7}	6.8×10^{-8}
0.4	0.23112	4.6×10^{-3}	2.3×10^{-4}	2.4×10^{-5}	3.6×10^{-6}	2.1×10^{-7}	4.2×10^{-8}
0.5	0.26319	3.0×10^{-3}	5.3×10^{-4}	1.0×10^{-5}	9.5×10^{-6}	8.1×10^{-8}	9.8×10^{-8}
0.6	0.29051	1.8×10^{-3}	9.6×10^{-4}	1.7×10^{-6}	1.2×10^{-5}	1.7×10^{-9}	1.3×10^{-7}
0.7	0.31418	9.4×10^{-4}	1.2×10^{-3}	3.4×10^{-6}	1.4×10^{-5}	4.6×10^{-8}	1.4×10^{-7}
0.8	0.33497	4.0×10^{-4}	1.3×10^{-3}	6.6×10^{-6}	1.4×10^{-5}	7.6×10^{-8}	1.5×10^{-7}
0.9	0.35345	3.5×10^{-5}	1.4×10^{-3}	8.6×10^{-6}	1.5×10^{-5}	9.4×10^{-8}	1.5×10^{-7}
1.0	0.37004	2.1×10^{-4}	1.4×10^{-3}	9.8×10^{-6}	1.4×10^{-5}	1.0×10^{-7}	1.4×10^{-7}
MAXE	-	2.1×10^{-2}	1.6×10^{-2}	3.7×10^{-4}	2.0×10^{-4}	4.0×10^{-6}	2.0×10^{-6}
AVGE	-	5.6×10^{-3}	2.9×10^{-3}	5.5×10^{-5}	2.8×10^{-5}	5.4×10^{-7}	2.7×10^{-7}

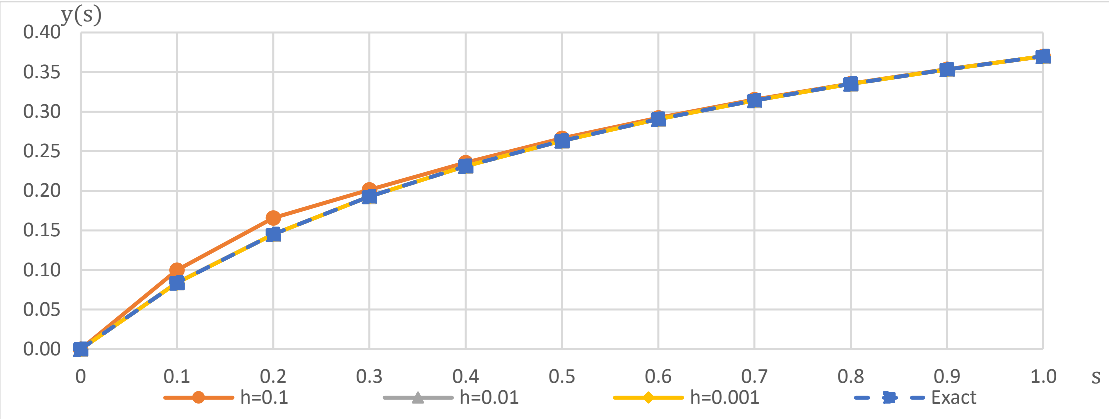

Figure 1: Behaviour of approximate solutions, $y(s)$ and point, s for Problem 1 using proposed method

Table 2: Numerical solutions for Problem 2 when $\alpha = 1.0$ at different step sizes, h

S	Exact	Absolute error at $h = 0.1$		Absolute error at $h = 0.01$		Absolute error at $h = 0.001$	
		FEM	TSLAB	FEM	TSLAB	FEM	TSLAB
0.0	0.00000	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}
0.1	0.10033	3.3×10^{-4}	3.3×10^{-4}	1.3×10^{-5}	7.1×10^{-6}	1.3×10^{-7}	8.4×10^{-8}
0.2	0.20271	1.7×10^{-3}	1.2×10^{-3}	2.8×10^{-5}	1.7×10^{-5}	2.9×10^{-7}	1.8×10^{-7}
0.3	0.30934	3.3×10^{-3}	2.2×10^{-3}	4.6×10^{-5}	2.9×10^{-5}	4.8×10^{-7}	3.0×10^{-7}
0.4	0.42279	5.3×10^{-3}	3.6×10^{-3}	7.1×10^{-5}	4.5×10^{-5}	7.3×10^{-7}	4.6×10^{-7}
0.5	0.54630	8.0×10^{-3}	5.5×10^{-3}	1.1×10^{-4}	6.8×10^{-5}	1.1×10^{-6}	6.9×10^{-7}
0.6	0.68414	1.2×10^{-2}	8.2×10^{-3}	1.6×10^{-4}	1.0×10^{-4}	1.7×10^{-6}	1.1×10^{-6}
0.7	0.84229	1.8×10^{-2}	1.2×10^{-2}	2.5×10^{-4}	1.6×10^{-4}	2.6×10^{-6}	1.6×10^{-6}
0.8	1.02960	2.8×10^{-2}	1.9×10^{-2}	3.9×10^{-4}	2.5×10^{-4}	4.1×10^{-6}	2.6×10^{-6}
0.9	1.26020	4.4×10^{-2}	3.1×10^{-2}	6.6×10^{-4}	4.2×10^{-4}	6.8×10^{-6}	4.3×10^{-6}
1.0	1.55740	7.3×10^{-2}	5.3×10^{-2}	1.2×10^{-3}	7.4×10^{-4}	1.2×10^{-5}	7.6×10^{-6}
MAXE	-	7.3×10^{-2}	5.3×10^{-2}	1.2×10^{-3}	7.4×10^{-4}	1.2×10^{-5}	7.6×10^{-6}
AVGE	-	1.9×10^{-2}	1.4×10^{-2}	2.3×10^{-4}	1.5×10^{-4}	2.3×10^{-6}	1.5×10^{-6}

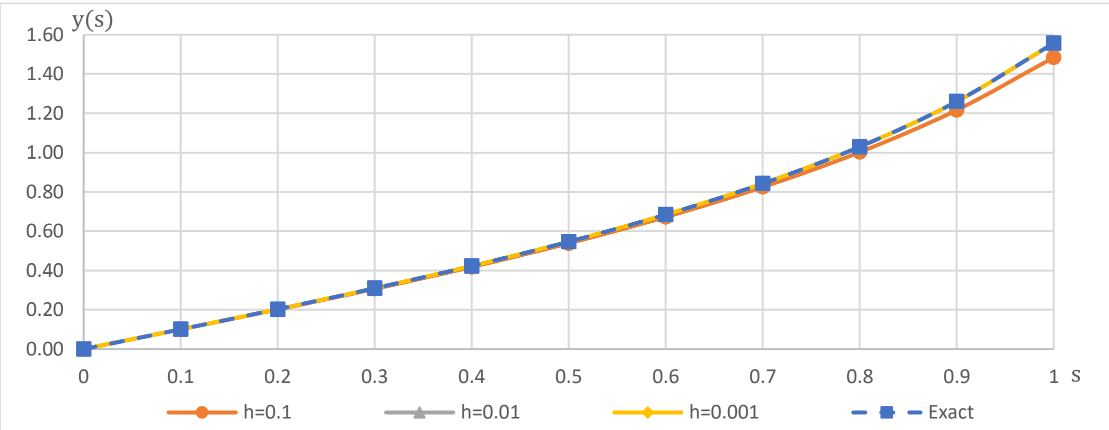

Figure 2: Behaviour of approximate solutions, $y(s)$ and point, s for Problem 2 using proposed method

Table 3: Numerical solutions for Problem 3 when $\alpha = 1.0$ at different step sizes, h

S	Exact	Absolute error at $h = 0.1$		Absolute error at $h = 0.01$		Absolute error at $h = 0.001$	
		FEM	TSLAB	FEM	TSLAB	FEM	TSLAB
0.0	0.00000	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}
0.1	0.099668	3.3×10^{-4}	3.3×10^{-4}	1.2×10^{-5}	7.7×10^{-6}	1.3×10^{-7}	8.1×10^{-8}
0.2	0.19738	1.6×10^{-3}	1.1×10^{-3}	2.4×10^{-5}	1.5×10^{-5}	2.5×10^{-7}	1.5×10^{-7}
0.3	0.29131	2.7×10^{-3}	1.8×10^{-3}	3.3×10^{-5}	2.1×10^{-5}	3.3×10^{-7}	2.1×10^{-7}
0.4	0.37995	3.5×10^{-3}	2.2×10^{-3}	3.9×10^{-5}	2.4×10^{-5}	3.9×10^{-7}	2.4×10^{-7}
0.5	0.46212	4.0×10^{-3}	2.4×10^{-3}	4.1×10^{-5}	2.5×10^{-5}	4.1×10^{-7}	2.5×10^{-7}
0.6	0.53705	4.1×10^{-3}	2.4×10^{-3}	3.9×10^{-5}	2.4×10^{-5}	3.9×10^{-7}	2.4×10^{-7}
0.7	0.60437	3.8×10^{-3}	2.3×10^{-3}	3.5×10^{-5}	2.2×10^{-5}	3.5×10^{-7}	2.2×10^{-7}
0.8	0.66404	3.3×10^{-3}	1.9×10^{-3}	3.0×10^{-5}	1.8×10^{-5}	2.9×10^{-7}	1.8×10^{-7}
0.9	0.71630	2.7×10^{-3}	1.6×10^{-3}	2.3×10^{-5}	1.4×10^{-5}	2.3×10^{-7}	1.4×10^{-7}
1.0	0.76159	2.1×10^{-3}	1.1×10^{-3}	1.6×10^{-5}	1.0×10^{-5}	1.6×10^{-7}	1.0×10^{-7}
MAXE	-	4.1×10^{-3}	2.4×10^{-3}	4.1×10^{-5}	2.5×10^{-5}	4.1×10^{-7}	2.5×10^{-7}
AVGE	-	2.8×10^{-3}	1.2×10^{-3}	2.9×10^{-5}	1.8×10^{-5}	2.9×10^{-7}	1.8×10^{-7}

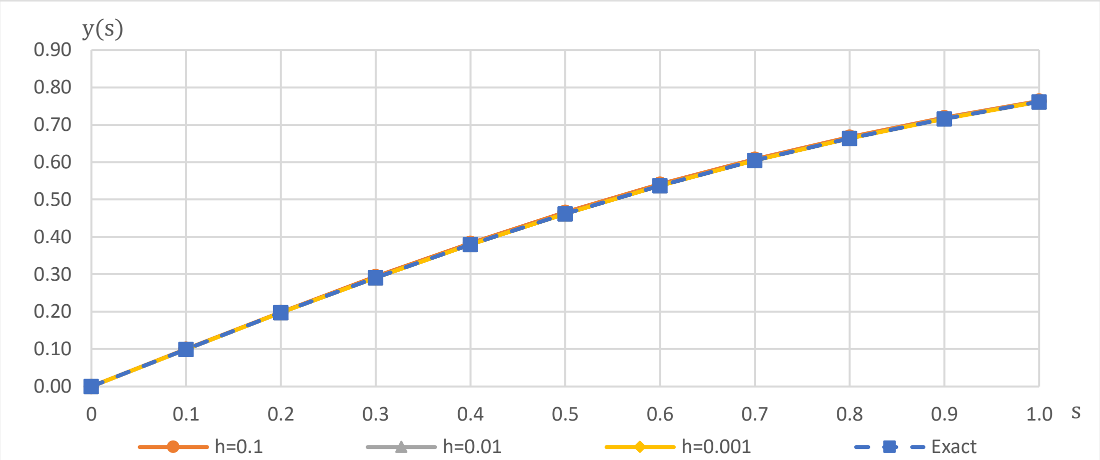

Figure 3: Behaviour of approximate solution, $y(s)$ and point, s for Example 3 using proposed method

Table 4 presents the numerical results for solving Example 4. The absolute error at each point for different values of α, where $\alpha = 0.95, 0.90$ and 0.80 is obtained when solving using the proposed method FEM and TSLAB at $h = 0.01$. The numerical result of FEM is comparable compared to TSLAB. As we can observed from Table 4, the absolute errors, average error, and maximum error are getting smaller when α is approaching to 1.00. Hence, the proposed method, FEM performed well in solving the problem when α is closer to 1.00.

6. Conclusion

In this article, FEM of order two is introduced where the step size $2h$ are considered in the derivation technique. According to the numerical results obtained, FEM is proved to be able to achieve comparable results compared to the existing methods, TSLAB, in each numerical example. In addition, the numerical results also validate the convergence analysis where the approximate solutions indeed converge as the step size, h is getting smaller. Besides, the numerical result also shows that better accuracy has yielded when the order of FDE, α increases and approaches to 1.00. Thus, the proposed method, FEM are reliable and appropriate to act as an alternative method to be implemented in solving different kinds of FDEs. The limitation of this proposed method is that it's not able to solve the problem with α that is far from 1.00. The future scope of this current study can be extended to increase the order of the method or solve fractional differential equations with delay.

Table 4: Numerical results for Example 4 at $h = 0.01$ at different α

S	Absolute error at $\alpha = 0.95$		Absolute error at $\alpha = 0.90$		Absolute error at $\alpha = 0.80$	
	FEM	TSLAB	FEM	TSLAB	FEM	TSLAB
0.0	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}	0.0×10^{0}
0.1	56×10^{-5}	3.3×10^{-5}	1.4×10^{-4}	1.0×10^{-4}	4.2×10^{-4}	4.0×10^{-4}
0.2	8.4×10^{-5}	7.1×10^{-5}	2.8×10^{-4}	2.7×10^{-4}	1.0×10^{-3}	1.1×10^{-3}
0.3	1.5×10^{-4}	1.5×10^{-4}	5.6×10^{-4}	5.8×10^{-4}	2.2×10^{-3}	2.4×10^{-3}
0.4	2.6×10^{-4}	2.6×10^{-4}	9.8×10^{-4}	1.0×10^{-3}	3.8×10^{-3}	4.1×10^{-3}
0.5	3.9×10^{-4}	4.0×10^{-4}	1.5×10^{-3}	1.6×10^{-3}	5.8×10^{-3}	6.2×10^{-3}
0.6	5.7×10^{-4}	5.8×10^{-4}	2.2×10^{-3}	2.3×10^{-3}	8.3×10^{-3}	8.8×10^{-3}
0.7	7.7×10^{-4}	7.9×10^{-4}	3.0×10^{-3}	3.1×10^{-3}	1.1×10^{-2}	1.2×10^{-2}
0.8	1.0×10^{-3}	1.0×10^{-3}	3.9×10^{-3}	4.0×10^{-3}	1.4×10^{-2}	1.5×10^{-2}
0.9	1.3×10^{-3}	1.3×10^{-3}	4.9×10^{-3}	5.1×10^{-3}	1.8×10^{-2}	1.9×10^{-2}
1.0	1.6×10^{-3}	1.6×10^{-3}	6.1×10^{-3}	6.3×10^{-3}	2.2×10^{-2}	2.3×10^{-2}
MAXE	1.6×10^{-3}	1.6×10^{-3}	1.7×10^{-3}	6.1×10^{-3}	2.3×10^{-2}	2.2×10^{-2}
AVGE	5.5×10^{-4}	5.5×10^{-4}	5.8×10^{-3}	2.1×10^{-3}	8.1×10^{-3}	7.7×10^{-3}

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