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Second-Order Accuracy in Time of Finite Difference Methods for Computational Aeroacoustics

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ABSTRACT

The recently developed second-order accuracy in time finite difference method suitable for computational aeroacoustics (CAA) is introduced. Although, it is straight forward to compute the coefficients for finite-difference method of any order of accuracy using the Taylor series and to then further optimize them to enhance their wavenumber preserving properties, there are difficult questions concerning their numerical stability The goal of this work is to develop an effective numerical technique that includes both linear and nonlinear wave propagation in order to solve acoustics problems in time and space. It also aims to evaluate the accuracy, effectiveness, and stability of the new technique. In 1-D linear and nonlinear computational aeroacoustics, the novel techniques were used. The findings of the conventional methods (square wave (FTCS) technique and step wave lax approach) are presented in this paper, and it is shown that the FTCS method is typically unstable for hyperbolic situations and cannot be employed. Unfortunately, the FTCS equation has very little practical application. It is an unstable method, which can be used only (if at all) to study waves for a short fraction of one oscillation period. Nonlinear instability and shock formation are thus somewhat controlled by numerical viscosity such as that discussed in connection with Lax method equation. The second-order accuracy in time finite difference method is more efficient than the (square wave (FTCS), step wave lax) methods and is faster than the step wave lax method.

Keywords: Numerical Method; finite difference; lax method; leapfrog; stability

INTRODUCTION

Aeroacoustics is a branch of acoustic science that studies noise generation and propagation. This has sundry applications in the aerodynamics and aircraft industries for predicting sound generated by the airframe, cavities, and the broadband noise generated by turbomachinery. Accurate noise prediction, which comes from an understanding of the underlying physics is primary for noise reduction (F. Q. Hu et al. 1996). Both computational and experimental studies are being conducted to uncover these mechanisms. However, experimental studies have problems with cost, safety, atmospheric variability, and reflection in wind tunnels. On the other hand, improvements in computer capability and numerical models promise accurate estimates at reasonable costs. This has led to the emergence of a new field: Computational Aeroacoustics (CAA) (Calvo, Franco & Rández 2004). CAA is a part of Aeroacoustics that focus on predicting the unsteady flow development and noise generation over complex geometries using high order numerical methods (Bayliss et al. 1985). In CAA, the full time-dependent compressible Navier-Stokes equations are solved numerically to simulate aerodynamic noise generation and propagation. Unlike conventional Computational Fluid Dynamics (CFD), any problem which CAA seeks to solve is almost by dentation time-dependent. This results in flow variables containing nonlinear waves across a wide range of frequencies. Resolving the highest frequency waves, which have extremely short wavelengths, becomes a formidable obstacle to accurate numerical simulation (Arguillat et al. 2010). Additionally, the amplitude of acoustic waves is much smaller than the flow, which demands high order numerical methods.

Flow disturbances have a tendency to degrade quickly away from a body or their source of origin for basic CFD problems. As a result, they barely affect the computational domain's boundaries. Contrarily, acoustic waves degrade relatively slowly and have a chance to reflect across the barrier into the computational domain, potentially contaminating the solution (Bull, M. K. 1996). Therefore, at the artificial exterior borders that allow the waves to escape smoothly, radiation and outflow boundary restrictions must be created. That is why unique CAA numerical methods have been under development in recent years (Colonius and Lele 2004).

APPLICATION OF AEROACOUSTICS NOISE

AEROACOUSTICS NOISE GENERATED BY A GENERIC SIDE VIEW CAR MIRROR

Away from a body or their source of creation, flow disturbances for ordinary CFD issues often dissipate very quickly. Therefore they have limited affect on the boundary of the computing domain's perimeter. The solution may be contaminated because acoustic waves, on the other hand, degrade very slowly and have a chance to reflect into the computational domain at the boundary (Bull, M. K. 1996). Therefore, radiation and outflow boundary conditions that allow the waves to escape smoothly must be imposed at the artificial external borders. In Figure 1 shows the flow around the side view car mirror. (Yong-Ju Chu 2018).

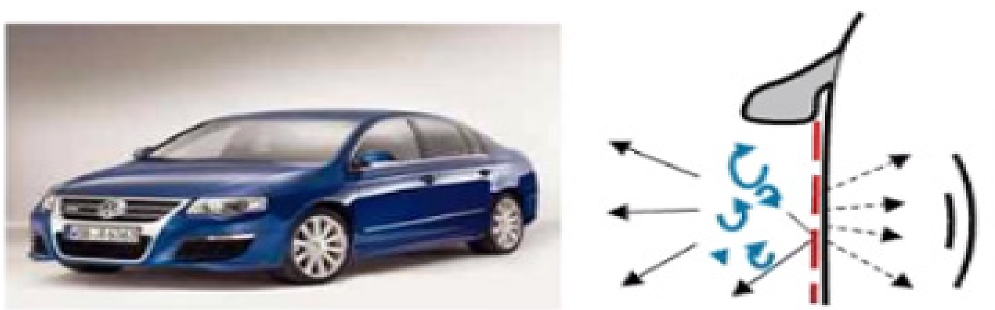


FIGURE 1. Flow around side view car mirror
source: Yong-Ju Chu (2018)

AEROACOUSTICS NOISE GENERATED BY FAN NOISE

One challenge in the automotive industry is to ensure that the noise level in an automotive cooling system or an air handling system is low enough for all operating conditions (Bogey, C. & Bailly, C. 2004). Thanks to the considerable progress made over the last decade on the motor noise, for most operating conditions, the blower has now become the major noise contributor flow for the cabin noise level for this reason, automobile manufacturers are placing increased emphasis on the reduction of the cabin noise level. This has resulted in more stringent noise requirements for the design of air handling systems and other cooling systems.

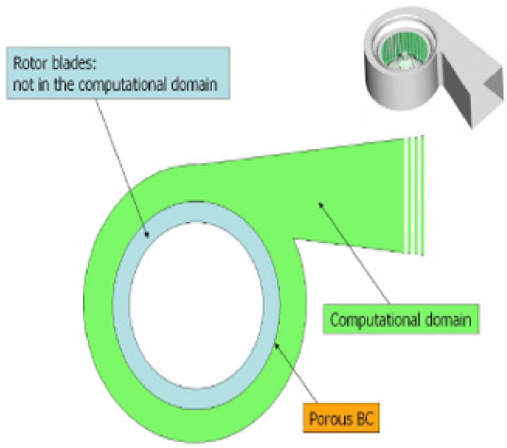


FIGURE 2. Fan noise is generated around and behind the blades
source: Stéphane et al. (2007)

AEROACOUSTICS NOISE GENERATED BY AIRCRAFT

A study by Freund et al. (1998), stated that an aircraft's noise is highly complicated and frequently comes from the engine, turbomachinery, fan, and jet exhaust. The engine fan and jet provide the propulsive noise, whereas all other aircraft structures contribute for the airframe noise. According to Curle (1955), there are four different types of aircraft noise: jet noise occurs when the exhaust's high velocity is mixed with the ambient air, combustor noise which is associated with the rapid oxidation of jet fuel and the associated release of energy, turbomachinery noise, which is heard when the source and the aircraft are close together, and aerodynamic noise, which is caused by the rapid air movement over the airframe and control surfaces. The greatest noise concern for aviation is still aerodynamic noise, but technological advancements in modern aircraft have resulted in significant reductions in combustor and turbomachinery noise (UK 2002).

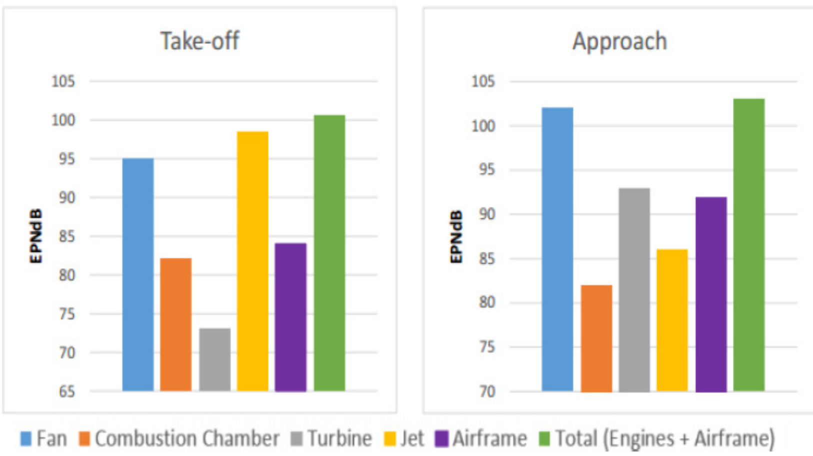


FIGURE 3. Breakdown of aircraft noise sources during take-off and landing

In order to reach the stringent requirements of CAA, high accuracy numerical method in both space and time are necessary to accurately simulate the linear and nonlinear propagation of the disturbances. The pure propagation of waves through a medium is by dentition linear, nonetheless, nonlinear effects do occur in many flows in the actual world, including internal thermoacoustic cooling fluxes, air turbulence, and sonic booms. There is a lack of efficient numerical techniques (Cockburn et al. 2000). The conventional numerical methods such as lax method, determinants of accuracy and stability. The objective of this work to construct an efficient numerical method for solving the acoustics problems in both time and space, to construct an efficient numerical technique for solving the acoustics problems and that contains both the linear and nonlinear propagation of the wave and to analyze the accuracy, efficiency and stability of the new method.

METHODOLOGY

In this section we first show the steps of the research method and then we selected two problems from the Workshop on Benchmark Problems in Computational Aeroacoustics (CAA) (Tam 1995). We displayed the derivation and analyze the flux-conservative initial value problems advection equation and the methods to solve Calvo et al. 2001). The first method is the forward time central space square-wave method (FTCS) and the second is the step wave lax method (Najafi-Yazdi et al. 2013). Following that analyze and derive the new method (staggered leapfrog second -order accurate in both time and space).

EXPERIMENT DESIGN AND PROCEDURE

1. Benchmark Problems in Computational Aeroacoustics (CAA) (Tam 1995), we use two problem as shown below:

a. Problem one

$$\frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} = 0$$

where

$$\Delta x = 1\,,\, C = 1$$

with the initial condition:

$$u(x,0) = u_0(x) = \{\quad 1 \qquad x \leq 0, x \geq 2\ 0 < x < 1 \quad x - 1 \quad 1 \leq x < 2 \qquad 0$$

b. Problem two

Linear wave equation

$$\frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} = 0$$

where:

$$\Delta x = 1\,,\, C = 1$$

and the initial condition:

$$u(x,0) = sin\, sin\ (5 - x)$$

$$u(5,t) = sin\, sin\ (\omega t)$$

$$5 \leq x \leq 450$$

The problem is investigated for the frequency of

$\omega = \frac{\pi}{4}$ at $t = 3200$.

For $t = 3200$s the exact solution is:

$$u(x,t) = sin\, sin\ \left(\omega(t - x + 5)\right) \qquad 5 \leq x + 5$$

2. From (Tam 1995) we take all properties, the values of the problem and the boundary condition.

3. Test the problem on the new method and get a new scheme.

4. Analysis the Stability (Von Neumann) for new scheme.

5. Compare the accuracy of the scheme with other conventional schemes.

METHODS TO SOLVE FLUX-CONSERVATIVE INITIAL VALUE PROBLEMS (ADVECTION EQUATION) AND VON NEUMANN STABILITY ANALYSIS

1. Forward Time Central Space (FTCS)

Square-Wave Method

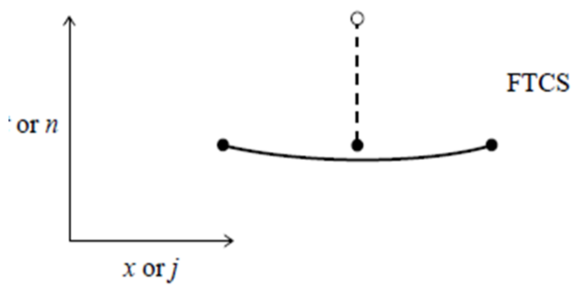


FIGURE 5. The Forward Time Centered Space differencing scheme

$$\frac{\partial u}{\partial t}\Big|_{j,n} = \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

The forward Euler is a first-order accuracy in $\Delta t \Delta t$.

$$\frac{\partial u}{\partial t}\Big|_{j,n} = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

The central Euler is only first order accurate in $\Delta x \Delta x$.

$$u_j^{n+1} = u_j^n - \frac{C\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

2. Step Wave Lax Method

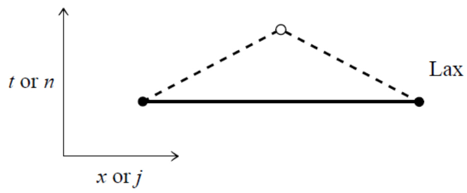


FIGURE 6. Representation of the Lax differencing

The FTCS method's instability can be fixed with a straightforward modification made possible by Lax. One changes with another u_j^n in the time derivative term by its average

$$u_j^n \rightarrow \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right)$$

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{c\,\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

The lax representation step wave method.

The von Neumann stability analysis of two methods above:

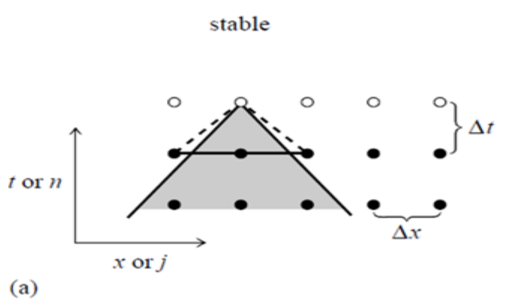


FIGURE 7. The von Neumann stability analysis of the lax method.

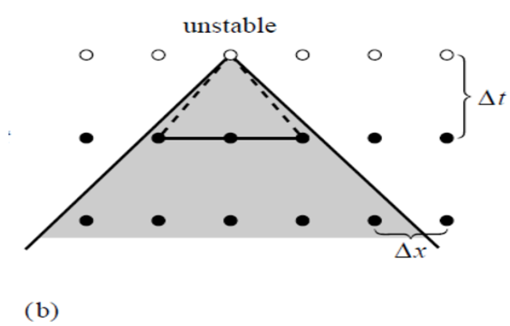


FIGURE 8. The von Neumann stability analysis of the (FTCS) method.

1. Second-Order Accuracy in Time Staggered Leapfrog Method

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^{n+1/2} - F_{j+1/2}^{n+1/2}\right)$$

$$u_j^{n+1} = u_j^n - \alpha\left[\frac{1}{2}\left(u_{j+1}^n + u_j^n\right) - \frac{1}{2}\alpha\left(u_{j+1}^n - u_j^n\right) - \frac{1}{2}\left(u_j^n + u_{j-1}^n\right) + \frac{1}{2}\alpha\left(u_j^n - u_{j-1}^n\right)\right]$$

The above representation second accuracy in time finite difference methods.

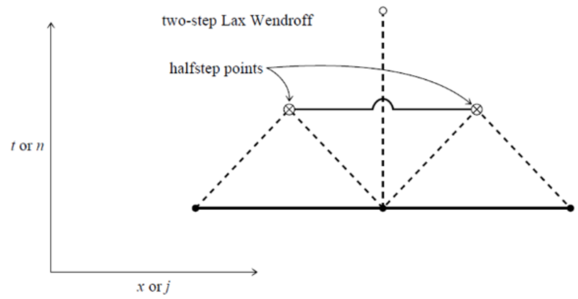


FIGURE 9. Representation of the second order accuracy in time differencing scheme

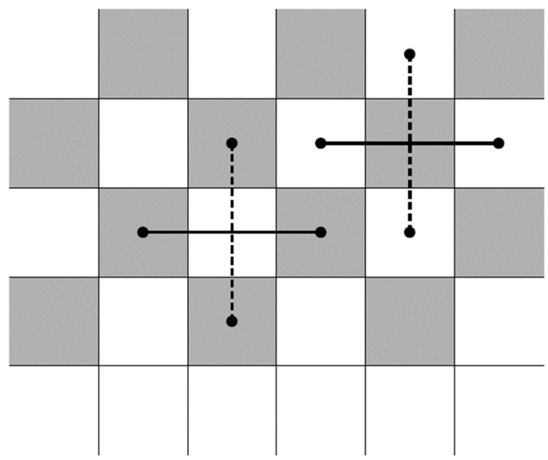


FIGURE 10. The Von Neumann stability analysis of the (FTCS) method

RESULTS AND DISCUSSION

1. Results of Problem One

1.1 Results of the FTCS differencing method

(a) CFL = 0.01

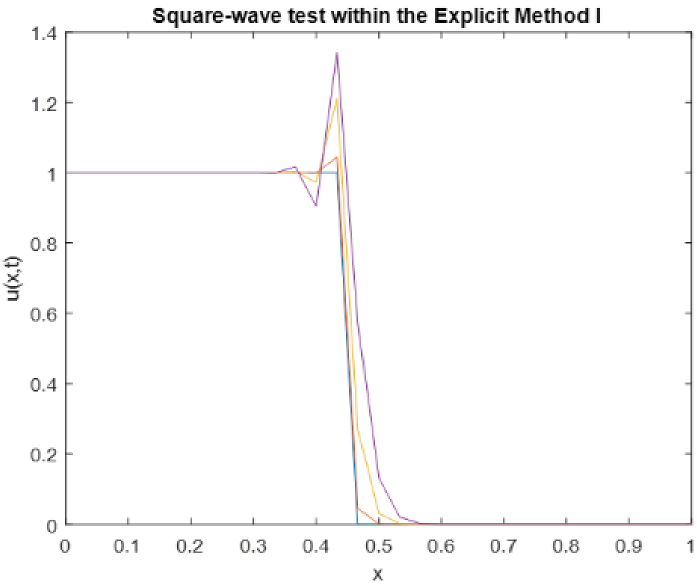


FIGURE 11. Square-wave (FTCS) method test within the explicit method with CFL = 0.01

(b) CFL = 0.5

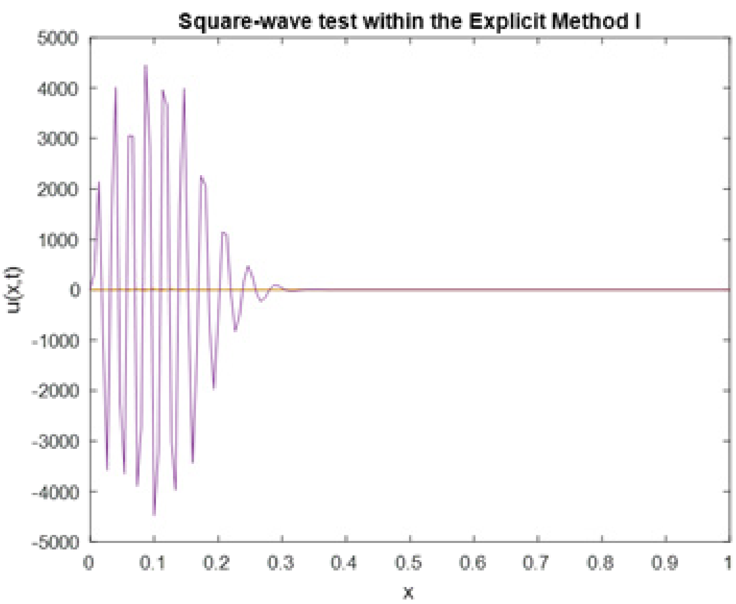


FIGURE 12. Square-wave (FTCS) method test within the explicit method with CFL = 0.5

(c) CFL = 1

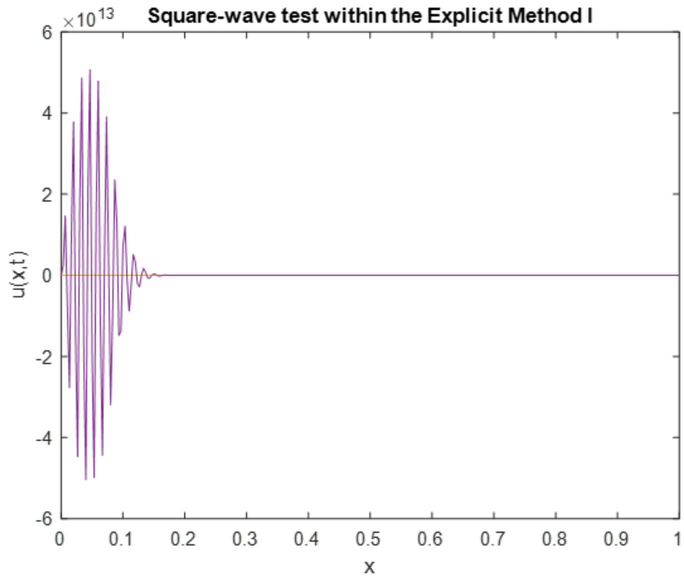
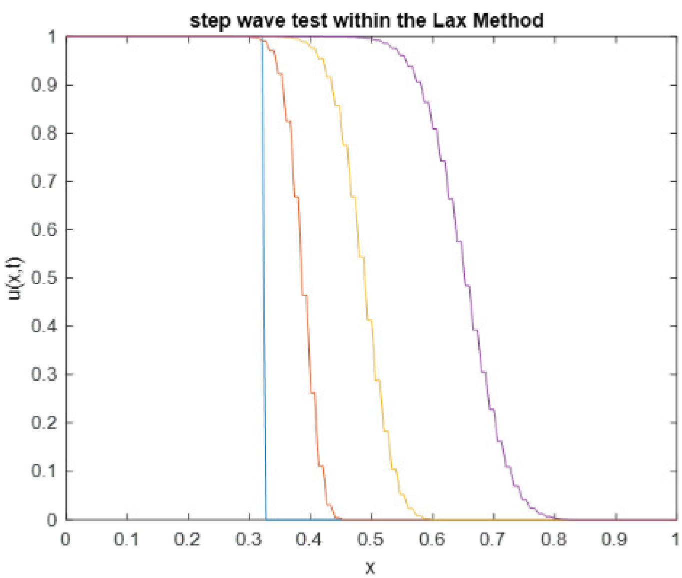


FIGURE 13. Square-wave (FTCS) method test within the explicit method with CFL = 1.

1.2 Results of Lax differencing step wave method

(a) CFL = 1



a) CFL=0.5

FIGURE 14. Step wave test within the Lax method with CFL = 0.5.

(b) CFL = 1

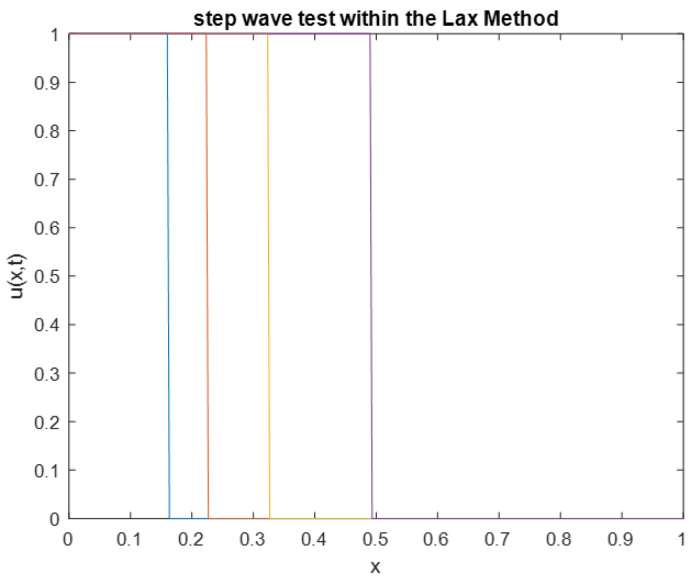


FIGURE 15. Step wave test within the Lax method with CFL = 1.

(c) CFL = 2

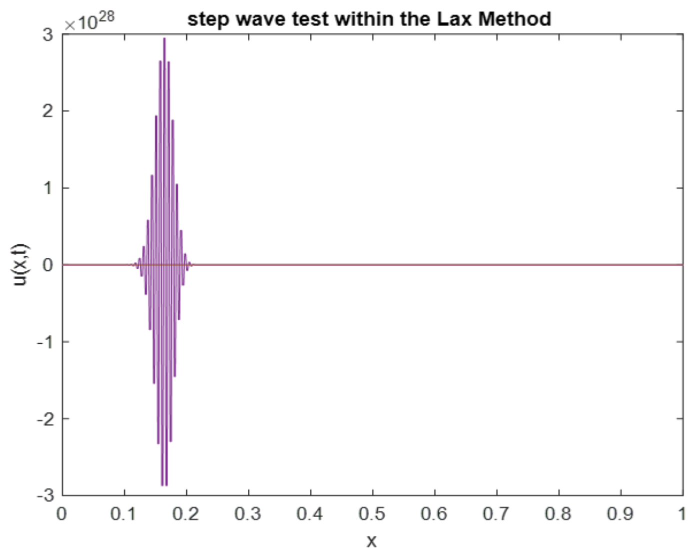


FIGURE 16. Step wave test within the Lax method with CFL = 2.

From the above results and after applying the von Neumann stability analysis we get

$$\xi = cosk\Delta x - i\frac{c\Delta t}{\Delta x}sink\Delta x$$

The stability condition $|\xi(k)| < 1$ leads to the requirement: CFL= $c\Delta t\Delta x \leq 1$. Step-wave Lax differencing must operate more quickly than square-wave FTCS. According to the aforementioned findings, any CFL values below one transform into the precise solution's diffuser solution. Greater than this value 1, however, causes the solution to become unstable. When it changed to CFL two, it was depicted in the figures. This approach can be summed up as follows:

1. If CFL= $\frac{c\Delta t}{\Delta x} > 1 \rightarrow$ the method gets unstable.
2. If CFL= $c\Delta t/\Delta x < 1 \rightarrow$ the method gets diffusive (it gets worse to get smaller time steps).

3. If CFL= cΔt/Δx=1→ the method converges to exact result.

1.3 Results of Staggered Leapfrog Differencing Method

(a) CFL = 0.5

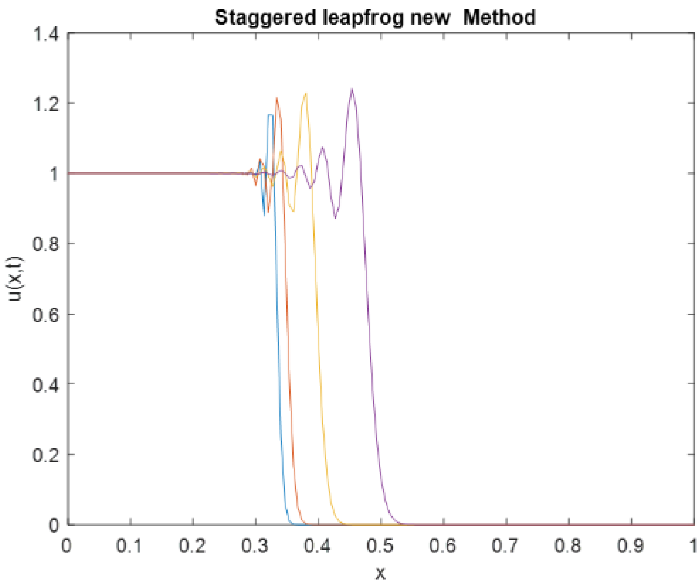


FIGURE 17. Staggered leapfrog method for CFL=0.5

(b) CFL = 1

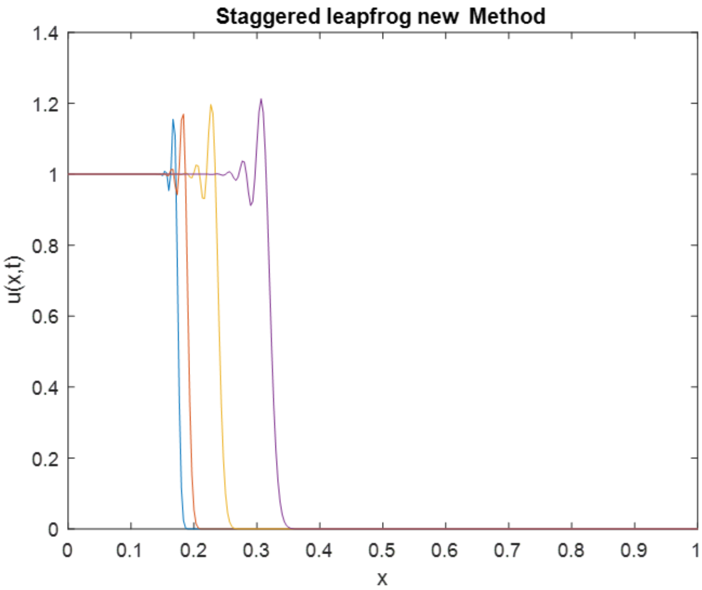


FIGURE 18. Staggered leapfrog method for CFL=1

(c) CFL = 2

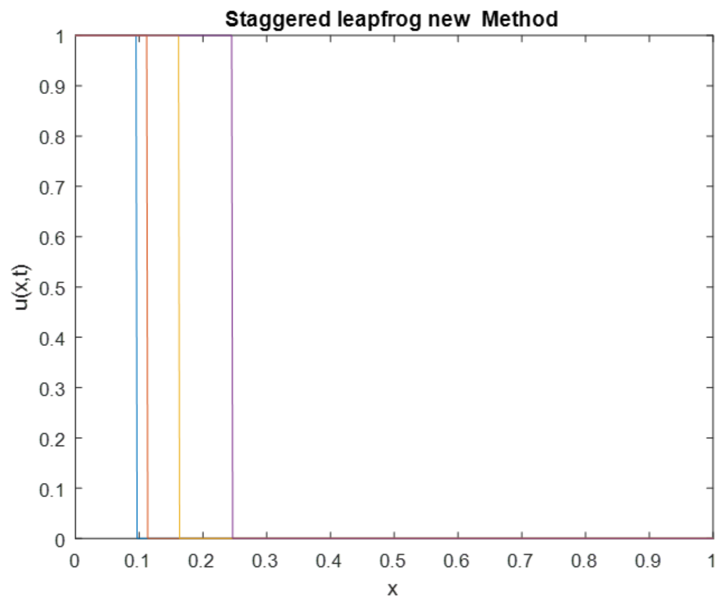


FIGURE 19. Staggered leapfrog method for CFL=2

(d) CFL = 2.2

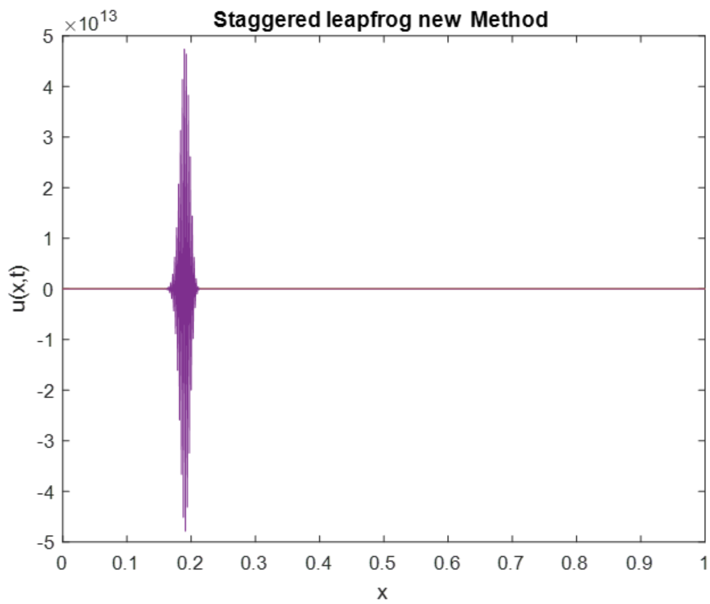


FIGURE 20. Staggered leapfrog method for CFL = 2.2

Following application of the von Neumann stability analysis, which now yields a quadratic equation for ξ(k) rather than a linear one, from the results of the staggered leapfrog approach shown above.

$$\xi^2 - 1 = -2i\xi \frac{c\Delta t}{\Delta x} sink\Delta x$$

$$\xi = -i\frac{c\Delta t}{\Delta x} sink\Delta x \pm \sqrt{1 - \left(\frac{c\Delta t}{\Delta x} sink\Delta x\right)^2}$$

Thus, the courant condition is again required for stability, in fact $|\xi(k)|= 1$ (no diffusion) for any $c\Delta t \leq \Delta x$. The previous method (square wave (FTCS), step wave Lax) is expensive and dangerous computationally. However, this staggered leapfrog method (second-order accurate in both space and time) can often be pushed right to their stability limit with correspondingly smaller computation times. The staggered leapfrog method is faster than square-wave (FTCS) and Lax differencing (step wave) method.

CONCLUSION

The main objective of this work was to develop and analyze numerical method (The second-order accuracy in time and space (staggered leapfrog) method) suitable for computational aeroacoustics (CAA). Although, it is straight forward to compute the coefficients for finite-difference method of any order of accuracy using the Taylor series and to then further optimize them to enhance their wavenumber preserving properties, there are difficult questions concerning their numerical stability. In this work show the results of the conventual method (square wave (FTCS) method, step wave lax method) and the FTCS method is generally unstable for hyperbolic problems and cannot usually be used. Unfortunately, the FTCS equation has very little practical application. It is an unstable method, which can be used only (if at all) to study waves for a short fraction of one oscillation period. Numerical viscosity, such as that discussed in connection with the Lax method equation, is thus somewhat responsible for controlling nonlinear instability and shock production. The square wave (FTCS) and step wave lax methods are less effective than the second-order accuracy in time and space (staggered leapfrog) method, which is also fasterThe staggered leapfrog approach, which achieves second-order accuracy in time and space, prevents significant numerical dissipation and mesh drifting (Gottlieb, D., and Turkel, E. 1976). The leapfrog method typically becomes unstable as the gradients are big for equations more complex than our simple model equation, especially nonlinear equations. Therefore, the accuracy of the stable and unstable methods is almost comparable.

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REFERENCES

A. Najafi-Yazdi, L. M. M. 2013. A low-dispersion and low-dissipation implicit Runge – Kutta scheme. *Journal of Computational Physics* 233: 315–323. doi:10.1016/j.jcp.2012.08.050

Arguillat, B., Ricot, D., Bailly, C. & Robert, G. 2010. Measured wavenumber: Frequency spectrum associated with acoustic and aerodynamic wall pressure fluctuations. *The Journal of the Acoustical Society of America* 128(4): 1647–1655. doi:10.1121/1.3478780

Bayliss, A., Parikh, P., Maestrello, L. and Turkel, E. 1985. Fourth Order Scheme for the Unsteady Compressible Navier-Stokes Equations. ICASE Report 85–44. doi:org/10.2514/6.1985-1694

Bull, M. K. 1996. Wall-pressure fluctuations beneath turbulent boundary layers: some reflections on forty years of research. *Journal of Sound and Vibration* 190(3): 299–315. doi:org/10.1063/1.3622773

Calvo, M., Franco, J. M. & Rández, L. 2004. A new minimum storage Runge-Kutta scheme for computational acoustics. *Journal of Computational Physics* 201(1): 1–12. doi:10.1016/j.jcp.2004.05.012

Calvo, M. P., Frutos, J. De & Novo, J. 2001. Linearly IMRK for advection reaction diffusion equations. *Applied Numerical Mathematics* 37: 535–549. doi:org/10.1155/2015/350362

Cockburn, B., Karniadakis, G. and Shu, C. W. 2000. *Discontinuous Galerkin Methods. Theory, Computational and Applications*. Berlin: Springer. doi:org/10.1002/zamm.200310088

Colonius, T. and Lele, S. K. 2004. Computational aeroacoustics: Progress in nonlinear problems of sound generation. *Progress in Aerospace Sciences* 40(6). doi:org/10.1016/j

Curle, N. 1955. The influence of solid boundaries upon aerodynamic sound. *Mathematical and Physical Sciences* 231(1187): 505–514. doi:org/10.1098/rspa.1955.0191

F. Q. Hu, M. Y. Hussaini, A. J. L. M. 1996. Low-Dissipation and Low-Dispersion Runge–Kutta Schemes for Computational Acoustics. *Journal of Computational Physics* 191(124): 177–191. doi:10.1006/jcph.1996.0052

Frank, H.M.; Munz, C.-D. 2016. Direct aeroacoustics simulation of acoustic feedback phenomena on a side-view mirror. *Vib, J. Sound* (371): 132–149.

Freund, J. B., Lele, S. K. and Moin, P. 1998. Direct Simulation of a Mach 1.92 jet and its sound field. AIAA/CEAS Paper 98–2291. doi:10.2514/6.1998-2291

Gottlieb, D. and Turkel, E. 1976. Dissipative Two-Four Method for Time Dependent Problems. *Mathematics of Computation* 30(136): 703–723. doi:10.2307/2005392