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A New Hybrid Non-standard Finite Difference-Adomian Scheme for Solution of Nonlinear Equations

(Skim Hibrid Baru Beza-terhingga Tak Piawai-Adomian bagi Penyelesaian Persamaan Tak Linear)

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ABSTRACT

This research develops a new non-standard scheme based on the Adomian decomposition method (ADM) to solve nonlinear equations. The ADM was adopted to solve the nonlinear differential equation resulting from the discretization of the differential equation. The new scheme does not need to linearize or non-locally linearize the nonlinear term of the differential equation. Two examples are given to demonstrate the efficiency of this scheme.

Keywords: Adomian decomposition method; Logistic equation; Lotka-Volterra system; non-standard schemes

ABSTRAK

Penyelidikan ini membangunkan satu skim tak piawai baru berdasarkan pada kaedah penguraian Adomian (KPA) bagi menyelesaikan persamaan tak linear. KPA ini diadaptasi untuk menyelesaikan persamaan tak linear yang terhasil daripada pendiskretan persamaan terbitan. Skim baru ini tidak perlu melinearkan atau melinearkan secara tak setempat sebutan tak linear persamaan terbitan itu. Dua contoh diberi untuk medemonstrasikan keefisienan skim ini.

Kata kunci: Kaedah penguraian Adomian; persamaan logistik; skim tak piawai; sistem Lotka-Volterra

INTRODUCTION

One of the shortcomings of the standard finite difference method is that the qualitative properties of the exact solution usually are not transferred to the numerical solution. Furthermore, many problems may affect the stability properties of the standard approach. Also, in practice, using the standard method the limit of the step-size is not reached. What we obtain is the numerical solution for one or several values of the step-size (Ibijola et al. 2008).

Non-standard finite difference schemes (NSFD) have emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential and biological models as well as chaotic systems (Mickens 2005). These techniques have many advantages over classical techniques and provide an efficient numerical solution. In fact, the non-standard finite difference method is an extension of the standard finite difference method. Non-standard schemes as introduced by Mickens (1989,1990,1994) are used to help resolve some of the issues related to numerical instabilities. Furthermore, Mickens (1999,2000,2005) introduced certain rules for obtaining the best difference equations, one of the most important of which is that the nonlinear terms of $f(t, y(t))$ are approximated in a non-local form.

If we do not linearize non-locally the system of differential equations, a somewhat better method is chosen - the Newton iteration method-to numerically solve the algebraic equation. This requires that $f(t, y(t))$ be smooth and that the inverse of the derivative operator f_y exist. Furthermore, to solve a system of equations, Newton's method might require a long time, so it is not economical. In this paper, a new non-standard finite difference scheme is presented in which the non-local linearization step is ignored when using the Adomian decomposition method (ADM) (Adomian 1988) to solve the nonlinear algebraic equations. The decomposition method yields rapidly convergent series solutions for both linear and nonlinear deterministic and stochastic equations (Cherruault & Adomian 1993). The technique has several advantages over the classical techniques, mainly it avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculations and avoidance of physically unrealistic assumptions. However, the ADM can be used to solve this problem effectively (Momani et al. 2006).

The rest of the paper is organized as follows. In the next section we present the non-standard finite difference method (NSFD) for solution of a system of first-order differential equations. Then we briefly describe ADM for systems of nonlinear algebraic equations. Next, we merge the NSFD and ADM to develop the non-standard scheme based on Adomian decomposition method to solve a system of nonlinear differential equations. Logistic equations and the Lotka-Volterra system are considered as test examples, and we discuss numerical approximations to the solutions. In the last section we summarize the conclusions.

NON-STANDARD FINITE DIFFERENCE METHOD

We seek to obtain the NSFD (Mickens 2000) solution for a system of differential equations of the form

$$y'_k = f(t, y_1, y_2, \ldots, y_m), \quad k = 1, 2, \ldots, m, \tag{1}$$

where $f(t, y_k(t))$ is the nonlinear term in the differential equation. Using the finite difference method we have

$$y'_1 = \frac{y_{1,k+1} - \psi_1(h)y_{1,k}}{\phi_1(h)}, \tag{2}$$

$$y'_2 = \frac{y_{2,k+1} - \psi_2(h)y_{2,k}}{\phi_2(h)}, \tag{3}$$

$$\vdots$$

$$y'_m = \frac{y_{m,k+1} - \psi_m(h)y_{m,k}}{\phi_m(h)}, \tag{4}$$

where ϕ_k and ψ_k are functions of the step size $h = \Delta t$. The ψ_k and ϕ_k have the following properties:

$$\psi_k = 1 + o(h), \tag{5}$$

$$\phi_k(h,\lambda) = h + o(h^2), \tag{6}$$

where $h \to 0$ and λ is fixed. The numerator functions ψ_k are usually equal to one (Lubuma & Patidar 2005), unless the system has dissipation.

Examples of functions $\phi_k(h, \lambda)$ that satisfy (6) are h, $\sin(h)$, $\sinh(h)$, $e^h - 1, \frac{1-e^{-\lambda h}}{\lambda}$.

Non-linear terms can in general be replaced by nonlocal discrete representations, for example,

$$y^2 \approx y_k y_{k+1}, \tag{7}$$

$$y^3 \approx \left(\frac{y_{k+1} + y_{k-1}}{2}\right) y_k^2, \tag{8}$$

Let $h = T/N$, $t_n = nh$, $n = 0, 1, \ldots, N \in Z^+$. Then (1) can be discretized as follows:

$$y_k(t_{n+1}) = \psi_k(h)y_k(t_n) + \phi_k(h)f(t_{n+1}, y_k(t_{n+1}), y_k(t_n)). \tag{9}$$

where $f(t_{n+1}, y_k(t_{n+1}), y_k(t_n))$ is the product the non-local linearization of $f(t_{n+1}, y_k(t_{n+1}))$.

ADOMIAN DECOMPOSITION METHOD (ADM)

The properties of ADM can found in (Vahidi et al. 2009). Consider the following system of nonlinear equations:

$$g_i(u_1, u_2, \ldots, u_n) = 0, \quad i = 1, 2, \ldots, n, \tag{10}$$

where $g_i : \Re^n \to \Re$. Equation (10) can be written in the form:

$$u_i = v_{i,0} + N_{i,0}(u_1, \ldots, u_n) \quad i = 1, 2, \ldots, n, \tag{11}$$

where $v_{i,0}$ are constants and $N_{i,0}$ are nonlinear operators. The decomposition method allows a solution to equations having the series form:

$$u_i = \sum_{j=0}^{\infty} u_{i,j}, \quad i = 1,2,\ldots,n. \tag{12}$$

The nonlinear operators $N_{i,0}$ are decomposed as an infinite series called Adomian polynomials:

$$N_{i,0}(u_1,\ldots,u_n) = \sum_{j=0}^{\infty} A_{i,j}, \quad i = 1,2,\ldots,n, \tag{13}$$

where $A_{i,j}$ depends upon $u_{1,0}, u_{1,1}, \ldots, u_{1,j}, u_{2,0}, u_{2,1}, \ldots, u_{2,j}, u_{n,1}, \ldots, u_{n,j}$. In view of (12) and (13) one has:

$$N_{i,0}\left(\sum_{j=0}^{\infty} \lambda^j u_{1,j}, \ldots, \sum_{j=0}^{\infty} \lambda^j u_{n,j}\right) = \sum_{j=0}^{\infty} A_{i,j} \quad i = 1,2,\ldots,n, \tag{14}$$

which yields:

$$A_{i,j} = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j} N_{i,0}\left(\sum_{j=0}^{\infty} \lambda^j u_{1,j,\ldots}, \sum_{j=0}^{\infty} \lambda^j u_{n,j}\right)\right]_{\lambda=0}, \tag{15}$$

where λ is the parameter introduced for convenience to represents. Therefore, (11) can be rewritten as:

$$\sum_{j=0}^{\infty} u_{i,j} = v_{i,0} + \sum_{j=0}^{\infty} A_{i,j}, \quad i = 1,2,\ldots,n. \tag{16}$$

The Adomian decomposition method identifies $u_{i,j}$, $j \geq 0$ by the following recursive relation:

$$v_{i,0} = v_{i,0}, \tag{17}$$

$$u_{i,j+1} = A_{i,j}, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots. \tag{18}$$

The solution $u_{i,j}$ can be approximated by the truncated series:

$$\varphi_{i,k} = \sum_{j=0}^{k-1} u_{i,j}, \tag{19}$$

such that:

$$\lim_{k \to \infty} \varphi_{i,k} = u_i, \quad i = 1,2,\ldots,n. \tag{20}$$

THE NUMERICAL ALGORITHM BASED ON ADOMIAN DECOMPOSITION

Applying the first derivatives developed by Mickens (1994) to the solution of a system of nonlinear differential equations in (1) gives:

$$\frac{y_{1,k+1} - y_{1,k}}{\phi_1(h)} = f(y_{1,k+1}), \tag{21}$$

$$\frac{y_{2,k+1} - y_{2,k}}{\phi_2(h)} = f(y_{2,k+1}), \tag{22}$$

$$\vdots$$

$$\frac{y_{m,k+1} - y_{m,k}}{\phi_m(h)} = f(y_{m,k+1}), \tag{23}$$

where the functions $\phi_k(h)$ satisfy the condition in (6). Solving the above system for $y_{i,k+1}$, $i = 1,2,\ldots,m$ yields

$$y_{1,k+1} = y_{1,k} + \phi_1(h) f(y_{1,k+1}), \tag{24}$$

$$y_{2,k+1} = y_{2,k} + \phi_2(h) f(y_{2,k+1}), \tag{25}$$

$$\vdots$$

$$y_{m,k+1} = y_{m,k} + \phi_m(h) f(y_{m,k+1}). \tag{26}$$

We now use ADM to solve this system of algebraic equations. Assume:

$$y_{j,k+1} = \sum_{i=0}^{\infty} u_i, \qquad j = 1,2,\ldots,m. \tag{27}$$

where:

$$u_0 = y_k, \tag{28}$$

$$u_1 = \phi_1(h) f(A_0), \tag{29}$$

$$u_2 = \phi_2(h) f(A_1), \tag{30}$$

$$\vdots$$

$$u_n = \phi_n(h) f(A_{n-1}). \tag{31}$$

For the n-term of the ADM solution we have

$$y_{j,k+1} = \sum_{i=0}^{n-1} u_i, \qquad j = 1,2,\ldots,m. \tag{32}$$

Application and Results

Solving a Logistic Equation Using NSFD-ADM

We use the NSFD-ADM technique to solve a logistic equation (Mickens 1994) of the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(1-y), \tag{33}$$

with the initial condition $y(0) = 0.5$. The exact solution of this equation is:

$$y(t) = \frac{0.5}{0.5(1+e^{-t})}. \tag{34}$$

The new scheme (NSFD-ADM) for solution of the logistic equation is:

$$\frac{y_{k+1} - y_k}{\phi(h)} = y_{k+1} - (y_{k+1})^2, \tag{35}$$

when $\phi(h) = 1 - e^{-h}$ (Mickens, 2006).

Solving Equation (35) for y_{k+1} gives:

$$y_{k+1} = y_k + \phi(h) y_{k+1} - \phi(h)(y_{k+1})^2. \tag{36}$$

Using ADM to solve Equation (36) yields:

$$y_{k+1} = \sum_{i=0}^{n-1} u_i, \tag{37}$$

where:

$$u_0 = y_k, \tag{38}$$

$$u_1 = \phi(h)(u_0 - u_0^2), \tag{39}$$

$$u_2 = \phi(h)(u_1 - 2u_0 \mathrm{u}_1), \tag{40}$$

$$\vdots$$

$$u_n = \phi(h)\left(u_{n-1} - \sum_{j=0}^{n-1} u_j u_{n-1-j}\right). \tag{41}$$

Solving the Lotka-Volterra System Using NSFD-ADM

The prey-predator differential equation system takes the form:

$$x' = ax - bxy, \tag{42}$$

$$y' = -cy + dxy, \tag{43}$$

where a,b,c,d are positive parameters. For the initial conditions $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$, all solutions except the fixed point at are periodic (Mickens, 2003).

In the Lotka-Volterra system, the values for the parameters (a,b,c,d) in the prey-predator system (42) and (43) are all taken to be one, yielding:

$$x' = x - xy, \tag{44}$$

$$y' = -y + xy. \tag{45}$$

We now illustrate Mickens's scheme (Mickens 2003) and our scheme to solve the Lotka-Volterra system.

1. The finite difference scheme proposed by Mickens (NSFD) to solve this system is to take:

$$\frac{x_{k+1} - x_k}{\phi(h)} = 2x_k - x_{k+1} - x_{k+1} y_k, \tag{46}$$

$$\frac{y_{k+1} - y_k}{\phi(h)} = -y_{k+1} + 2x_{k+1} y_k - x_{k+1} y_{k+1}, \tag{47}$$

where $\phi(h) = \sin h$.

2. The new non-standard scheme incorporating Adomian decomposition (NSFD-ADM) is to take:

$$\frac{x_{k+1} - x_k}{\phi(h)} = x_{k+1} - x_{k+1} y_{k+1}, \tag{48}$$

$$\frac{y_{k+1} - y_k}{\phi(h)} = -y_{k+1} + x_{k+1} y_{k+1}, \tag{49}$$

where $\phi(h) = \sin h$. Solving (48) and (49) for x_{k+1} and y_{k+1}, respectively, giving:

$$x_{k+1} = x_k + \phi(h)(x_{k+1} - x_{k+1}y_{k+1}), \quad (50)$$

$$y_{k+1} = y_k + \phi(h)(-y_{k+1} + x_{k+1}y_{k+1}). \quad (51)$$

Using ADM to solve (50) and (51) yields

$$x_{k+1} = \sum_{i=0}^{n-1} u_i, \quad y_{k+1} = \sum_{i=0}^{n-1} v_i, \quad (52)$$

where:

$$u_0 = x_k, \quad v_0 = y_k, \quad (53)$$

$$u_1 = \phi(h)(u_0 - u_0v_0), \quad v_1 = \phi(h)(-v_0 + u_0v_0), \quad (54)$$

$$\vdots$$

$$u_n = \phi(h)\left(u_{n-1} - \sum_{i=0}^{n-1} u_j v_{n-1-j}\right), \quad (55)$$

$$v_n = \phi(h)\left(-v_{n-1} + \sum_{j=0}^{n-1} u_j v_{n-1-j}\right). \quad (56)$$

Results and Discussion

The Lotka-Volterra system in (44) and (45) was numerically integrated using the NSFD-ADM scheme as coded in the computer algebra package Maple. In Maple, the number of variable digits controlling the number of significant digits is set to 35. In all the calculations done in this paper, we set the parameters (a,b,c,d) of the Lotka-Volterra system equal to one with initial conditions $x(0) = 20$, $y(0) = 1$.

Table 1 shows the accuracy of the NSFD-ADM for solution of the logistic equation. Comparing NSFD-ADM rersults with the exact solution, we see that the maximum difference between the NSFD-ADM solution and the exact solution at time steps $\Delta t = 0.05$ and $\Delta t = 0.01$ is of the order of magnitude of 10^{-6}. Thus we can conclude that the NSFD-ADM solutions for the time step $\Delta t = 0.01$ is sufficiently accurate for our comparison purposes.

TABLE 1. Absolute errors between the exact solution and the 6-term NSFD-ADM solution for the logistic equation

t	$\lvert Exact–NSED–ADM_{0.05}\rvert$	$\lvert Exact–NSFD–ADM_{0.01}\rvert$
1	5.975E-03	1.214E-03
2	7.450E-03	1.502E-03
3	5.372E-03	1.066E-03
4	3.035E-03	5.913E-04
5	1.519E-03	2.902E-04
6	7.110 E-04	1.332E-04
7	3.195E-04	5.868E-05
8	1.398 E-04	2.516E-05
9	5.998E-05	1.058E-05
10	2.536E-05	4.383E-06

In Table 2 we present the absolute errors between NSFD and NSFD-ADM solutions and the fourth-order Runge-Kutta method (RK4) solutions at time step $\Delta t = 0.01$ for the Lotka-Volterra system. It was found that the NSFD-ADM solutions agree very well with the RK4 solutions for t up to $t = 20$. We note that increasing the number of terms improves the accuracy of the NSFD-ADM solutions.

TABLE 2. Differences between the RK4 solution for the Lotka-Volterra system and solutions obtained using 6-term NSFD-ADM and NSFD

t	$\Delta = \lvert RK4_{0.01} – NSFD_{0.01}\rvert$		$\Delta = \lvert RK4_{0.01} – NSFD–ADM_{0}\rvert$	
	Δx	Δy	Δx	Δy
2	1.510E-06	1.635E-02	4.285E-07	4.201E-02
4	4.001E-07	7.449E-03	1.229E-07	5.291E-04
6	1.813E-06	1.724E-03	5.848E-07	6.333E-04
8	1.217E-05	3.312E-04	4.117E-06	1.821E-04
10	8.623E-05	5.831E-05	3.058e-05	3.785E-05
12	6.153E-04	9.821E-06	2.289E-04	6.959E-06
14	4.395E-03	1.679E-06	1.714E-03	1.225E-06
16	3.139E-02	3.610E-07	1.284E-02	2.380E-07
18	0.224	1.662E-07	9.62E-02	7.972E-08
20	1.6	3.496E-07	0.720	9.166E-08

The x -y phase portraits obtained using the NSFD-ADM, NSFD and RK4 solutions at $\Delta t = 0.01$ for the Lotka-Volterra system are shown in Figure 1. In this case, we see that the NSFD-ADM has the advantage over the NSFD on achieving a good accuracy with different time steps.

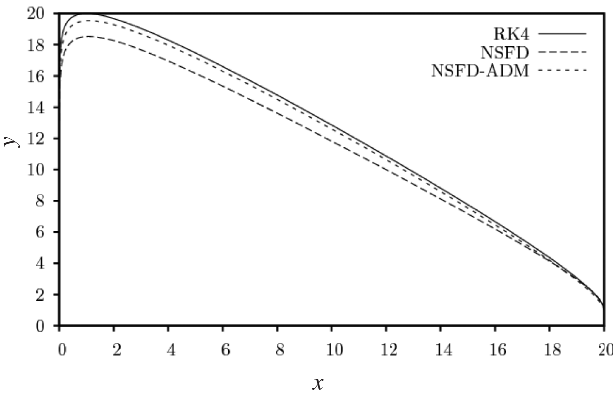

FIGURE 1. Numerical solution for the Lotka-Volterra system using RK4, NSFD and 6-term NSFD-ADM where $h = 0.01$, $x(0) = 20$ and $y(0) = 1$

In Figure 2, x_k and y_k are shown to be periodic, and the corresponding x_k and y_k phase-space curves were closed for the Lotka-Volterra system with initial conditions $x(0) = 0.1$, $y(0) = 1$ for time step $\Delta t = 0.001$.

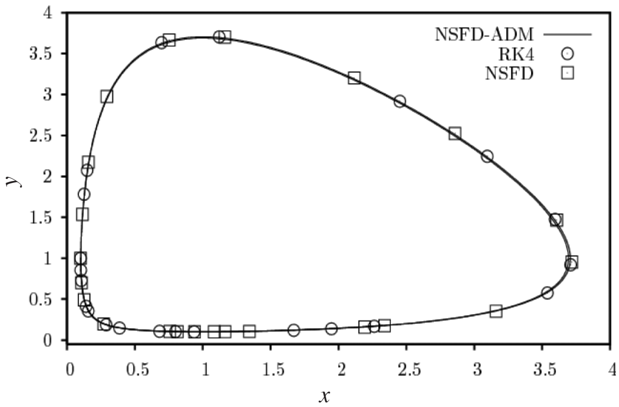

FIGURE 2. Numerical solution for the Lotka-Volterra system using RK4, NSFD and 6-term NSFD-ADM where $h = 0.001$, $x(0) = 0.1$ and $y(0) = 1$

CONCLUSIONS

In this paper, we derived a hybrid NSFD-ADM algorithm for nonlinear differential equations. Non-local linearization procedure of the nonlinear terms in NSFD was replaced with the Adomian polynomials. Solutions to the logistic equation and Lotka-Volterra system were presented to demonstrate the efficiency of the new scheme.

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