Jurnal Kejuruteraan 36(1) 2024: 199-209
https://doi.org/10.17576/jkukm-2024-36(1)-19

General Exact Analytical Expressions for Rotation and Displacement of a Timoshenko Beam Under Variable Loads with Validation

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Received 29 October 2023, Received in revised form 30 November 2023
Accepted 30 December 2023, Available online 30 January 2024

ABSTRACT

The researchers involved in the field of applied mathematics mostly project the efficient and viable solutions of those problems in which have practical applications in science and engineering. The Timoshenko beam model (TBM) is described by a system of ordinary differential equations where the expressions of rotation and displacement of the beam are ultimately required. Most of the work on analytical solutions of beam problems in literature focuses the elastic and Euler-Bernoulli beams, whereas for the TBM the numerical solutions are usually preferred. The analytical expressions for the TBM exist for only very basic load cases and are also not general for any load function. In this paper, we attempt to suggest a novel protocol based on expressing a load function by a polynomial or its power series development, and then use it to develop general analytical expressions for the rotation and displacement of a fixed TBM which is not load specific. The proposed general equations can provide ease of access and handling for the practitioners working in applied mathematics, structural engineering and mechanical vibrations as the developed equations can be used with only constant inputs to tailor particular expressions of rotation and displacement profiles of a fixed TBM under any variable load. For performance evaluations of the proposed general equations, we have also obtained particular expressions for some important variable loads, like: linearly varying loads ((LVLs): triangular and trapezoidal and quadratically varying loads (QVLs): parabolic/square and circular loads. Finally, the proposed protocol for generalization has been validated for fixed elastic beams under uniformly distributed loads (UDLs), and the results match exactly with those expressions available in literature. The contributions of this study, on one hand provide ready, direct and exact general expressions for the rotation and displacement profiles of a fixed TBM, while on the other hand the solution of such problem is achieved with quite negligible computational overhead, execution time and software implementations.

Keywords: Linearly varying loads; quadratically varying loads; Fixed Timoshenko Beam Model; general analytical expressions; rotation; deflection; variable load

INTRODUCTION

Beam theory play very important role in our daily life applications. Beams are elements which resist an applied load. So many engineers and scientists have worked on this theory and evolved different techniques to find out deflections and rotations in a beam subjected to distinct loads. Stephen Timoshenko, a Ukrainian-born scientist, devised a new beam theory at the turn of the twentieth century (Shaikh and Cheng, 2012). This idea is known as the Timoshenko beam (TB) theory after his name. The TB model took into account shear deformation as well as rotational inertia.

As a result, TB defines the behavior of a variety of beams, including short beams, composite sandwiched beams, and beams that can be driven at high frequencies as a result, the wavelength of excitation shortens and approaches the beam thickness. Researchers commonly employ a numerical approximation of the TB model as a starting point to acquire a better understanding of the Reissner-Mindlin problem, which is more complex. When these problems are handled using the finite difference method or normal Galerkin finite methods, a negative behavior known as locking phenomenon occurs (Timoshenko 1921). Small parameters cause this locking phenomena, and academics have proposed many strategies that should be uniform in terms of small parameters. For example, Loula, Hughes and Franca (1987) suggested a formulation that is linked to Petrov-Galerkin, while, Cheng, Han and Huang (1997), Cheng and Xue (2002), and Arnold (1981) proposed a mixed formulation that uses decreased integration to produce approximations. For the TB model, the authors suggested finite difference techniques in Arnold (1981). Cheng and Xue (2002) discussed the usage of the least-squares finite element approach. Arnold (1981) also, looked at the finite element method's p and h-p versions for the TB model. Researchers have also discovered a precise analytical solution to the Timoshenko beam problem for both uniform and continuous loads by Malik, Shaikh and Shaikh (2021a). It is imperative to mention some types of beam to tailor the discussion towards Timoshenko beam. In the following sub-sections, beams are classified on the basis of whether they are supported at ends or elsewhere. Beam which is fixed at one end and free at other end is called a Cantilever beam. A beam which is supported freely at the both ends is called simply supported beam. When the end portions of a beam are extended beyond the support then it is called over hanging beam. A beam whose both ends are fixed is called fixed beam. A beam which is provided more than two supports is said to be a continuous beam. These types are graphically summarized in Figure 1 (Website Link-I, 2023a). Similarly, the type of applied loads have influence on the rotation and displacement profiles of a beam. A point load, also known as a concentrated load, acts at a single point only on the beam. A uniformly or constantly distributed load (UDL) is one that is spread along the length of a beam in such a way that the rate of loading remains constant throughout the beam's distribution length. A uniformly changing load or variable load (UVL) is one that is spread along the span of the beam in such a way that the rate of loading does not remain constant from point to point throughout the beam's distribution length. Further in these loads, we have linearly varying loads (LVL), which are triangular and trapezoidal. Also, the parabolic/square load and circular load, both also known as quadratically varying (QVL), are uniformly varying loads. Figure 2 shows concentrated load, UDL, UVL (triangular) (Website Link-II, 2023b).

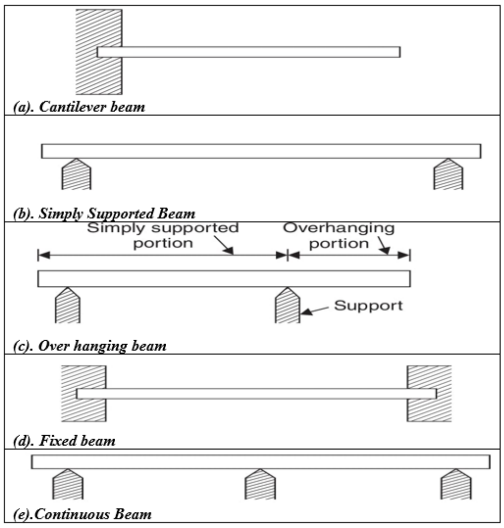

FIGURE 1. Types of beams due to placement of supports (Website Link-I, 2023a)

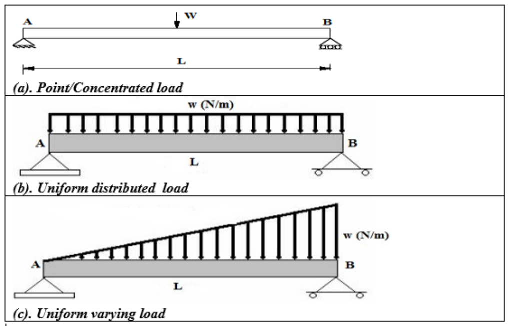

FIGURE 2. Types of loads (Website Link-II, 2023b)

The well-known Euler-Bernoulli beam (EBB) theory is particular case of TB theory for finding load-carrying and deflection characters of a beam. In EBB model, the beam has no change in angle of cross section about neutral line before and after deflection. The Timoshenko beam (TB) study was enlarged by S. Timoshenko and P. Ehrenfest early 20^{th} century (Loula, Hughes and Franca, 1987). TB model includes both shear deformation as well as rotational bending effects. In the case of TB model, beam is thick and angle of cross section about neutral line will change after deflection.

In TB theory, shear deformations are considered, whereas in EBB theory such deformations are not considered. In TB theory, the plane sections remain plane, but they are no longer normal to the longitudinal axis. In EBBs, the plane sections remain normal and parallel to the longitudinal axis. The TB model is better for beams with a low aspect ratio. In Figure 3 (Hibbeler, 2004; Hibbeler, 2005; Website Link-III, 2023c) the EBB and TB are compared. In EBB area of cross section and the neutral axis always form 90^0 angle and there is no rotation when the beam is subjected to loads. On the other hand in TB, when load is applied then besides examining the deflection and slope of bending, due to shear deformation in the beam we also consider rotation profile.

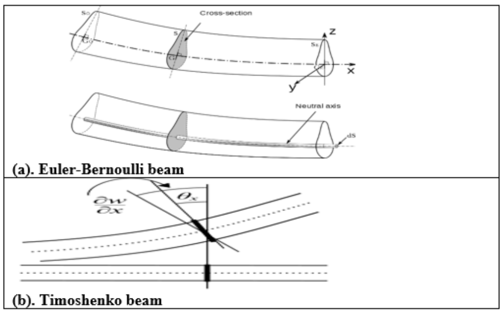

FIGURE 3. Comparison of EBB and TB (Website Link-III, 2023c)

There have been enormous works in the past on TBM. We include opinions of different scholars about solution of TB equations by using different numerical schemes to find the approximate rotation and deflection. But, it is observed that fewer have solved TBM analytically.

Shaikh and Cheng (2012) solved TB problem along with boundary conditions numerically by two non-standard finite difference schemes and locking phenomena was overcome due to uniform meshes. The solutions by approximate schemes matched quite well with the exact solutions. Malik, Shaikh and Shaikh (2021a) developed analytical technique, solved TB problem including boundary conditions independent of locking phenomena, while two loads were taken, first constant and other one variable. The results were validated with the previous studies on the similar cases. Malik, Shaikh and Shaikh (2021b) proposed and applied finite difference scheme to obtain numerical solution of Timoshenko beam under constant as well as variable load without facing locking phenomena and discretized system into algebraic sum. The results were obtained using MATLAB and agreed with the previous attempts on the similar cases.

Li (1990) considered a discretization of TB problem through the use of p and h-p versions of finite element method and error was reduced while locking phenomena vanished as the thickness of beam decreased. Chen et al. (2021) proposed an effective computational method to solve Timoshenko beam problem under complex load. This method was a modification of inverse finite element method and showed more accurate results when tested on a thin-walled aluminum beam under four different loads.

Mansoori, Torabi and Totonch (2020) founded different characters of beams using numerical schemes. Solved simply supported TB model under uniform load using FEM. Friedman and Kosmatka (1993), based on Hamilton's principle, created stiffness, mass and force matrices for a simple two-node system Element of a TB. The transverse and rotational displacements were represented by Lagrange polynomials, which were made interdependent by requiring them to fulfil Timoshenko's beam theory's two homogeneous differential equations. A short beam's displacement can be reliably predicted using the current element. According to numerical results, it predicts shear and moment resultants as well as natural frequencies better than existing finite elements when subjected to any sophisticated distributed stress using only one element. Wang (1995) provided the deflection and stress-resultant relationships for single-span TB and EBB beams under any transverse loading condition. These connections made it easier for engineers to calculate the deflection and stress resultants using the well-known EBB solutions. Thus, without the requirement for a more extensive f1exural-shear-deformation analysis, the influence of transverse shearing strain on the deflection and stress resultants may be easily accommodated.

Jelenić, Gordan and Edita (2011) established a method to find out exact solution of TB model using Lagrange's interpolation polynomial with finite possible nodal points. With sufficient internal nodes it was possible to get exact solution. Ghannadiasl, Amin and Mofid (2015) used the dynamic green function to show the free vibration of an elastically constrained Timoshenko beam on a partly Winkler basis. For modelling beam structures with diverse boundary conditions, an accurate and direct modelling technique was presented. Due to the Green function results were precise in closed forms. So this technique was more efficient and accurate. Davis, Henshell and Warburton (1972) derived stiffness matrices and consistent mass matrices. They performed tests of convergence for both simply supported beam and cantilever beam. When correct value of shear coefficient was used solution converged onto the exact solution. They also calculated accurate frequencies for the portal frame. It was concluded that Bernoulli-Euler and TB theories were unsatisfactory when depth and distance between leg joints had a same order. Wang (2008) based on Eringen's nonlocal elasticity theory and the Timoshenko beam theory, presented a variation consistent derivation of the governing equations and boundary conditions for beam bending. When dealing with micro and nano beams that are short and stubby, this nonlocal Timoshenko theory accounts for both the scale effect and the effect of transverse shear.

In summary, it has been observed that most of the recent work on the TB model was done numerically using the finite difference and finite element methods which are time consuming and computationally lengthy procedures. Whereas, a little focus was devoted on the analytical solutions of the TB model. The work of Malik, Shaikh and

Shaikh (2021a) was devoted to analytical solution of the TB model, but the attempts were load specific for a few types of loads. Hence, to the best of knowledge acquired through the literature survey, there has not been any attempt to derive general analytical solution for the deflection and rotation profile of the TB model which is valid any type of load.

Since, most of analytical work has been carried on EB beam theory and exact solutions which are load specific are only available for EB beam theory. Also, several authors have focused on numerical schemes for the solution of TB model. No one has obtained direct solutions in generalised form in case of TB model which are not load specific, rather all attempts have been load specific. So that's why, in this work, the general analytical solution of the TB model is focused and for a fixed case initially which will not be load specific. The proposed protocol is based on generalized form of applied load. We can determine rotation and deflection parameters without locking phenomena at once, and then just by some simplifications rather than applying techniques of integration or any other transformations at all each time, the results can be directly obtained.

DIFFERENTIAL EQUATION MODEL OF THE FIXED TBM

The ordinary differential equations which represent mathematical model of the Timoshenko beam problem (Shaikh and Cheng, 2012) while, $X \in (0, L)$, L being the length of the beam is described using the equations (1)-(4).

$$-\frac{dQ}{dX} = p \tag{1}$$

$$\frac{dM}{dX} = Q \tag{2}$$

$$\frac{d\theta}{dX} = -\frac{M}{EI} \quad (or) \quad -EI\frac{d^2\theta}{dX^2} - Q = 0 \tag{3}$$

$$-\frac{Q}{kGA} + \frac{dW}{dX} - \theta = 0 \tag{4}$$

Where, $p(X)$ is applied load, $M(X)$ is bending moment, $Q(X)$ is shear force, $\theta(X)$ is rotation of cross section, $W(X)$ is deflection of beam and k is correction factor for shear, G is the shear modulus, A is the cross-section area, EI is flexural rigidity, E is Young's modulus and I is second moment of intertia. If the beam is fixed at both ends, then the boundary conditions can be given as:

$$W(0) = W(L) = 0, \theta(0) = \theta(L) = 0$$

These mean that there is no deflection and rotation at both ends of the beam due to fixed supports.

Substituting in system of equations (1)-(4), to overcome the difficulties of unit system and conversion, the relations:

$$X = xL$$

$$Q = \frac{EI\sigma}{L^2},$$

$$\frac{pL^3}{EI} = f,$$

$$W = wL$$

we have system of ordinary differential equations in non-dimensionalized form, which is given in equations (5)-(7) as:

$$-\sigma' = f \tag{5}$$

$$-\theta'' - \sigma = 0 \tag{6}$$

$$-\sigma\varepsilon^2 + w' - \theta = 0 \tag{7}$$

Where $\varepsilon^2 = \frac{EI}{kGAL^2}$, ande ε is a parameter depending on the physical constants in the actual model. Thus, controlling ε, we can get the dimensionalized solutions through (5)-(7). The boundary conditions can now be stated as:

$$w(0) = w(L) = 0, \theta(0) = \theta(L) = 0$$

EXACT SOLUTION OF TBM UNDER SPECIFIC LOADS

The existing approach for finding exact load specific solution of the TB model in non-dimensionalized form as in (5)-(7) was usually obtained in literature and focused in texts. The idea is to consider specific varying load function $f(x)$ and then successively integrate the equations (6)-(7) and use through forward substitution successively. Thus, in this way one can get in the end rotation and deflection profiles: $\theta(x)$ and $w(x)$.

We consider four specific loads here: triangular, trapezoidal, parabolic/square and circular. These loads are mathematically described as follows:

Triangular load: $f_I(x) = 100x$

Trapezoidal load: $f_{II}(x) = 10 + 100x$

Parabolic/square load: $f_{III}(x) = 100x^2$

Circular load: $f_{IV}(x) = 100(4-x)(4+x)$

Where we consider: $0 \le x \le 1$.

For formal solution, for example we work out with the parabolic load. Substituting the parabolic load in (5), we have: $\sigma' = -100x^2$, and integrating throughout with regards to the space variable x leads to:

$$\sigma(x) = -\frac{100}{3}\ x^3 + c_1 \qquad (8)$$

Using (8) in (6), and integrating twice with respect to x leads to:

$$\theta(x) = \frac{100}{60}\ x^5 - \frac{c_1 x^2}{2} + c_2 x + c_3 \qquad (9)$$

Using $\theta(0) = 0$ in (9) gives: $c_3 = 0$, so we have:

$$\theta(x) = \frac{100}{60}\ x^5 - \frac{c_1 x^2}{2} + c_2 x \qquad (10)$$

Using $\theta(1) = 0$ in (10) leads to:

$$-c_1 + 2c_2 = -\frac{10}{3} \qquad (11)$$

Using (10) and (8) in (7) gives:

$$w'(x) = \frac{100}{60}\ x^5 + c_1\left(\varepsilon^2 - \frac{x^2}{2}\right) + c_2 x + -\frac{100\varepsilon^2}{3}\ x^3$$

Integrating this with respect to x we have:

$$w(x) = \frac{100}{360}\ x^6 + c_1\left(x\varepsilon^2 - \frac{x^3}{6}\right) + c_2\frac{x^2}{2} - \frac{100\varepsilon^2}{12}\ x^4 + c_4 \qquad (12)$$

Using $w(0) = 0$ in (12) we get $c_4 = 0$, so (12) becomes:

$$w(x) = \frac{100}{360}\ x^6 + c_1\left(x\varepsilon^2 - \frac{x^3}{6}\right) + c_2\frac{x^2}{2} - \frac{100\varepsilon^2}{12}\ x^4 \qquad (13) \qquad (13)$$

Using $w(1) = 0$ in (13) we have:

$$c_1(6\varepsilon^2 - 1) + 3c_2 = 50\varepsilon^2 - \frac{5}{3} \qquad (14)$$

Solving (11) and (14) simultaneously, we obtain:

$$c_2 = \frac{5(18\varepsilon^2 + 1)}{3(1 + 12\varepsilon^2)},\ c_1 = \frac{20\ (1 + 15\varepsilon^2)}{3\ (1 + 12\varepsilon^2)}$$

Finally using these constants in (10) and (13), we have the rotation and deflection profiles when the fixed TB is subjected to a parabolic load. Which are given in (15) and (16), respectively.

$$\theta(x) = \frac{5}{3}\ x^5 - \frac{10(15\varepsilon^2+1)}{3(12\varepsilon^2+1)}x^2 + \frac{5(1+18\varepsilon^2)}{3(12\varepsilon^2+1)}x \qquad (15)$$

$$w(x) = \frac{5}{18}\ x^6 - \frac{100\varepsilon^2}{12}\ x^4 - \frac{10(15\varepsilon^2+1)}{9(12\varepsilon^2+1)}x^3 + \frac{5(1+18\varepsilon^2)}{6(12\varepsilon^2+1)}x^2 + \frac{20(15\varepsilon^2+1)\varepsilon^2}{3(12\varepsilon^2+1)}x \qquad (16)$$

The rotation and deflection profiles in other types of loads can be obtain in the same way. These are mentioned in Table 1.

TABLE 1. Exact expressions for rotation and displacement of fixed TB with L=1 under various load cases I-IV

Load case	Rotation profile $\theta(x)$	Displacement profile $w(x)w(x)$
I	$\frac{25x^4}{6} - \frac{5(40\varepsilon^2+3)}{2(1+12\varepsilon^2)}x^2 + \frac{10(15\varepsilon^2+1)}{3(1+12\varepsilon^2)}x$	$\frac{5x^5}{6} - \frac{1}{6}\left(\frac{5(40\varepsilon^2+3)}{(1+12\varepsilon^2)} + 100\varepsilon^2\right)x^3 + \frac{5(15\varepsilon^2+1)}{3(1+12\varepsilon^2)}x^2 + \frac{5(40\varepsilon^2+3)}{(1+12\varepsilon^2)}\varepsilon^2 x$
II	$\frac{25}{6}x^4 + 5\frac{x^3}{3} - \frac{10(1+13\varepsilon^2)}{(1+12\varepsilon^2)}x^2 + \frac{5(5+72\varepsilon^2)}{6(1+12\varepsilon^2)}x$	$5\frac{x^4}{12} + \frac{5}{6}x^5 - \left(\frac{10(1+13\varepsilon^2)}{3(1+12\varepsilon^2)} + \frac{50\varepsilon^2}{3}\right)x^3 + 5\left(\frac{(5+72\varepsilon^2)}{12(1+12\varepsilon^2)} - \varepsilon^2\right)x^2 + \frac{20(1+13\varepsilon^2)\varepsilon^2}{(1+12\varepsilon^2)}x$
III	$\frac{100}{60}x^5 - \frac{20(15\varepsilon^2+1)}{3(12\varepsilon^2+1)}\frac{x^2}{2} + \frac{5(1+18\varepsilon^2)}{3(12\varepsilon^2+1)}x$	$\frac{5}{18}x^6 - \frac{100\varepsilon^2}{12}x^4 - \frac{10(15\varepsilon^2+1)}{9(12\varepsilon^2+1)}x^3 + \frac{5(1+18\varepsilon^2)}{6(12\varepsilon^2+1)}x^2 + \frac{20(15\varepsilon^2+1)\varepsilon^2}{3(12\varepsilon^2+1)}x$

continue ...

... cont.

IV	$800\frac{x^3}{3}-5\frac{x^5}{3}-\frac{10(1425\varepsilon^2+119)}{3(1+12\varepsilon^2)}x^2+\frac{5(942\varepsilon^2+79)}{3(1+12\varepsilon^2)}x$	$-5\frac{x^6}{18}+(200+25\varepsilon^2)\frac{x^4}{3}-\frac{10(1425\varepsilon^2+119)}{9(1+12\varepsilon^2)}x^3+\left(\frac{5(942\varepsilon^2+79)}{6(1+12\varepsilon^2)}-800\varepsilon^2\right)x^2+\frac{20(1425\varepsilon^2+119)\varepsilon^2}{3(1+12\varepsilon^2)}x$

PROPOSED PROTOCOL FOR GENERALIZATION

Looking at the most widely used cases of loads as discussed mathematically, we arrive at the observation that in real the load functions are usually single-piece polynomial functions. Also, the load functions are always thoroughly or sectionally continuous and integrable as demanded by the model (5)-(7) to accommodate integrations. Even if the load functions are not polynomials, these can be expressed in power series development about $x = 0$, which is also the Machlaurin's series. Hence, we can safely consider a generalized expression of the load function which becomes cross grounds to the developed general analytical expressions for the rotation and deflection profiles of a fixed TBM in this study. We assume the following expression of a generalized polynomial load functions:

$$f(x) = \sum_{i=0}^{n} a_i x^i \tag{17}$$

Equation (17) can also be used when the load function is not expressed mathematically but in form of experimental or sampled points through measurements. In such cases, the data can be interpolated first, leading to a polynomial again in form of (17) to accommodate general solutions. In the case of piece-wise load functions, the proposed protocol can be used in each individual piece, and finally the full resolution profiles can be analyzed altogether. Using (17) in the fixed TBM (5)-(7) can carrying out integrations in the similar way, we claim the following general analytical expressions for the rotation and deflection profiles of a fixed TBM which are not load specific, but these are adaptable for any load function. The expressions are given in (18) and (19) with specific form of constants in (20) and (21).

$$\begin{aligned} \theta(x) &= \sum_{i=0}^{n+3}\frac{a_i x^{i+3}}{(i+1)(i+2)(i+3)}-c_1\left(\frac{x^2}{2}\right)+c_2 x \\ w(x) &= \sum_{i=0}^{n+4}\frac{a_i x^{i+4}}{(i+1)(i+2)(i+3)(i+4)}+c_1\left(x\varepsilon^2-\frac{x^3}{6}\right) \end{aligned} \tag{18}$$

$$+c_2\frac{x^2}{2}-\sum_{i=0}^{n+2}\frac{a_i x^{i+2}}{(i+1)(i+2)}\varepsilon^2 \tag{19}$$

$$\begin{aligned} c_1 &= \sum_{i=0}^{n+3}\frac{a_i (L)^{i+3}}{(i+1)(i+2)(i+3)(L^2+12\varepsilon^2)} \\ &+12\sum_{i=0}^{n+2}\frac{a_i L^{i+1}}{(i+1)(i+2)(L^2+12\varepsilon^2)}\varepsilon^2 \\ &-12\sum_{i=0}^{n+4}\frac{a_i L^{i+3}}{(i+1)(i+2)(i+3)(i+4)(L^2+12\varepsilon^2)} \end{aligned}$$

$$\begin{aligned} c_2 &= -\frac{2(6\varepsilon^2-L^2)}{12\varepsilon^2+L^2}\sum_{i=0}^{n+3}\frac{a_i (L)^{i+2}}{(i+1)(i+2)(i+3)} \\ &+6\sum_{i=0}^{n+2}\frac{a_i L^{i+2}}{(i+1)(i+2)(12\varepsilon^2+L^2)}\varepsilon^2 \end{aligned} \tag{20}$$

$$-6\sum_{i=0}^{n+4}\frac{a_i L^{i+4}}{(i+1)(i+2)(i+3)(i+4)(12\varepsilon^2+L^2)} \tag{21}$$

It should be noted that the main objective of this study was to attain general analytical solution for the rotation and displacement profiles of a fixed TB which is not load specific. So, through (18)-(21) we conclude with the general analytical expressions for the rotation and displacement of the fixed TB model. Therefore, finally we can further analyze the behavior of beam under different types of loads using proposed general equations with less computational efforts by considering different values of ε which is sole dependent parameter of the problem. It should also be noted that in the above equations:

$a_i's$ are co-efficient of applied load.
c_1 and c_2 are arbitary constants of solution.
L is length of beam. n is degree of applied load polynomial function.

Therefore, we can easily get direct exact solution of any fixed TB which is subject to any polynomial load without having to repeat the integrations and all the process again and again as discussed in previous section for parabolic load.

PARTICULAR EXPRESSIONS, DISCUSSION AND VALIDATION

In this section, we have displayed displacement and rotation parameters of Timoshenko beam under four different types of loads by choosing five different values of $\varepsilon\varepsilon$, which is sole dependent parameter of the problem. Also, it is shown that the particular rotation and displacement profiles attained from our proposed general analytical solution for the three varying cases of loads match with those discussed in before using conventional technique. In the last, we also discuss validation of the proposed approach used to derive general analytical solution of the TB model on a fixed elastic beam with uniformly distributed or constant load, as in literature.

For the purpose of verification, we specify the conditions for different load cases in Table 2.

TABLE 2. Specific constants for different loads

Load case	n	a_0	a_1	a_2	$a_3, \ldots$	L
Triangular	1	0	100	0	0	1
Trapezoidal	1	10	100	0	0	1
Parabolic	2	0	0	100	0	1
Circular	2	1600	0	−100	0	1

The specific values in Table 2 used in general equations (18)-(21) successfully lead to the same particular expressions described in Table 1 for a fixed TBM under four considered loads.

To analyze the behavior of particular solutions in relation to different load functions and the physical parameter ε, we display the particular expressions of the rotation and displacement of the considered fixed TBM with unit length. In addition two graphs, one using plot command for the rotation and other by using semilogy command for the deflection in more detail using MATLAB have been presented for each load function. Figures 3, 5, 7, 9 represent he rotations in the fixed TBM for triangular, trapezoidal, parabolic and circular loads, respectively. Figures 4, 6, 8, 10 represent he deflections in the fixed TBM for triangular, trapezoidal, parabolic and circular loads, respectively. The effect of ε has been analyzed in each figure. It is, in general, observed that increasing ε, both deflection and rotation also increase. In view of figures 3, 4 we observe that the beam is subject to slight rotations in the range: -0.5 to 0.6 for all values of ε, whereas deflection profile exhibits positive values always. However, for higher values of ε, the deflection of the beam becomes unstable due to higher magnitude displacements. The maximum deflection occurs between 0.5 and 0.7 in the fixed TBM under triangular load. In view of figures 5, 6 we observe that the beam is subject to slight rotations in the range: -0.5 to 0.62 for all values of ε, whereas deflection profile exhibits positive values always. However, for higher values of ε, the deflection of the beam becomes unstable due to higher magnitude displacements. The maximum deflection occurs between 0.5 and 0.7 in the fixed TBM under trapezoidal load. The deflection tends to 80000 for $\varepsilon = 100$, indicating a sufficiently critical and sensitive case.

In view of figures 7, 8 we observe that the beam is subject to slight rotations in the range: -0.3 to 0.4 for all values of ε, whereas deflection profile exhibits positive values always. Since the load is sqaure, so deflections are mostly higher around the right end of the beam. However, for higher values of ε, the deflection of the beam becomes unstable due to higher magnitude displacements. The maximum deflection occurs around 0.6 in the fixed TBM under parabolic load. The deflection in this case is also sensitive to the values of parameter ε. In view of figures 9, 10 we observe that the beam is subject to quite higher rotations in the range: -13 to 13 for all values of ε, whereas deflection profile exhibits positive values always. The rotations for all considered values of the parameter ε are very close to each other as apparent from figure 9. Since the load is circular, so deflection is highest at the middle of the beam as expected. In reference to all figures 3-10, we observe the sensitivity of the TBM for values of parameter ε. For larger values of ε, the fixed TBM is subject to unstable regimes which confirms the practically adopted standard to keep ε to at most 1.

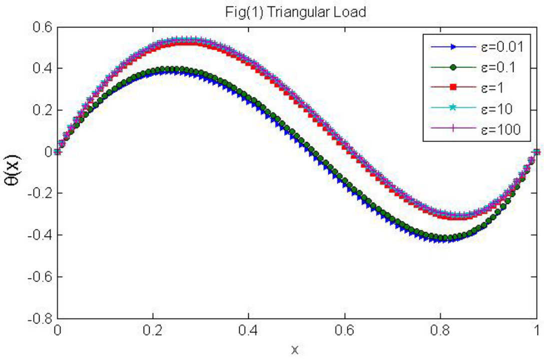

FIGURE 3. $\theta(x)$ of TBM for triangular load function

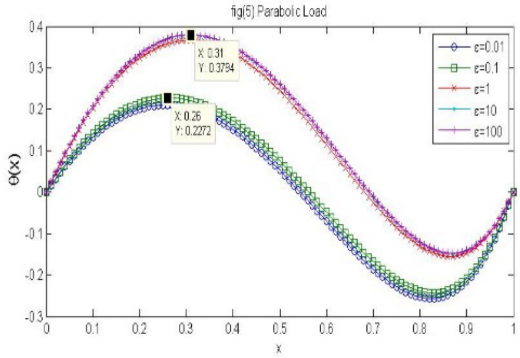

FIGURE 7. $\theta(x)$ of TBM for parabolic load function

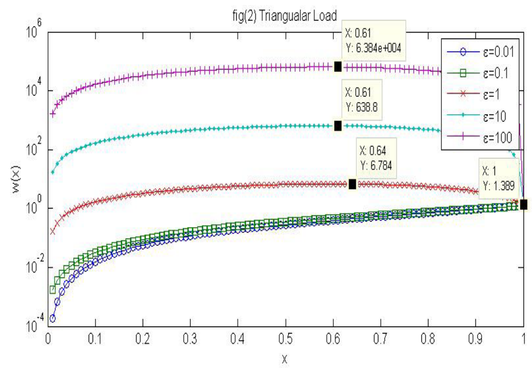

FIGURE 4. $w(x)$ of TBM for triangular load function

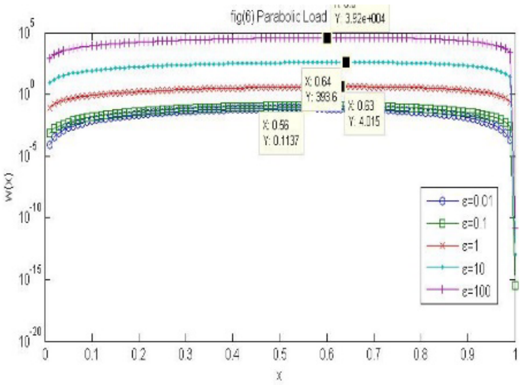

FIGURE 8. $w(x)$ of TBM for parabolic load function

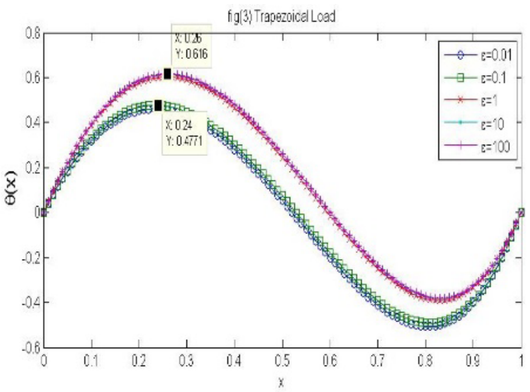

FIGURE 5. $\theta(x)$ of TBM for trapezoidal load function

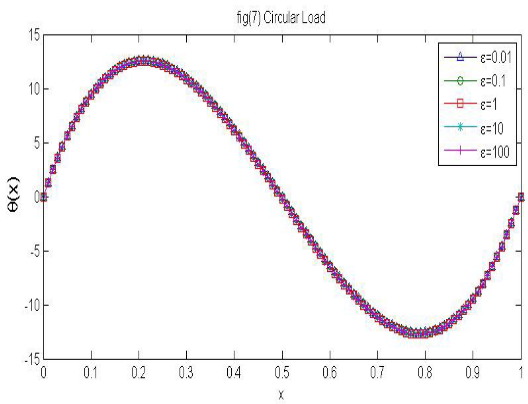

FIGURE 9. $\theta(x)$ of TBM for circular load function

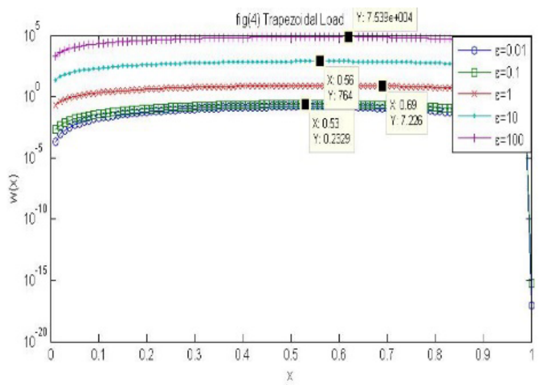

FIGURE 6. $w(x)$ of TBM for trapezoidal load function

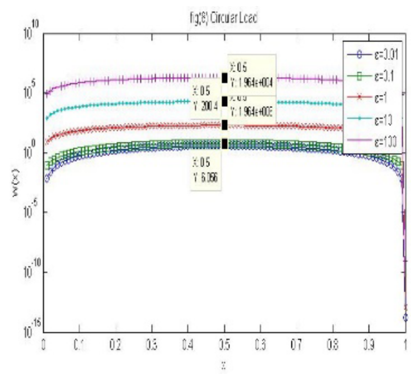

FIGURE 10. $w(x)$ of TBM for circular load function

VALIDATION

It is evident from the literature review presented in the introduction of this paper that most of the work was devoted to the existence and uniqueness of the solutions of the differential equation models of elastic beams, like the Euler-Bernoulli beams. However, for the case of TBM the particular expressions of the solution are mostly validated through the corresponding cases with elastic beams. We note that in the absence of shear deformation, the governing equations of the TBM reduce to that of elastic beams. For the purpose of verification of the proposed protocol in this study, we apply the generalized load function to attain the generalized expressions of the delfection and slope of an elastic beam model and compare the particular expressions for the case of UDL with those available in standard texts. The elastic beam differential equations are:

$$\frac{d^2M}{dx^2} = f(x) \tag{22}$$

$$\frac{ds}{dx} = \frac{M}{EI} \tag{23}$$

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \tag{24}$$

Where f is the applied load, M is bending moment, s is the slope of bending, EI is flexural rigidity and v is the deflection in the elastic beam. For the fixed case the displacement and the slope are zero at both ends of the beam. It is imperative to understand that the rotation effect of TBM is not considered in (22)-(24). Using The generalized load function as devised earlier for the TBM, we tailor the equations (22)-(24) towards the general analytical expressions of the deflection and bending slopes, which are given in (25)-(26). The corresponding coefficients are mentioned in (27)-(28).

$$v(x) = \frac{1}{EI}\left\{\sum_{i=0}^{n+4}\frac{a_i x^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right)\right\} \tag{25}$$

$$s(x) = \frac{1}{EI}\left\{\sum_{i=0}^{n+3}\frac{a_i x^{i+3}}{(i+1)(i+2)(i+3)} + c_1\left(\frac{x^2}{2}\right) + c_2 x\right\} \tag{26}$$

$$c_1 = 12\sum_{i=0}^{n+4}\frac{a_i L^{i+1}}{(i+1)(i+2)(i+3)(i+4)} - 6\sum_{i=0}^{n+3}\frac{a_i L^{i+1}}{(i+1)(i+2)(i+3)} \tag{27}$$

$$c_2 = 2\sum_{i=0}^{n+3}\frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)} - 6\sum_{i=0}^{n+4}\frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)(i+4)} \tag{28}$$

In the case of a UDL, say when $f(x) = a$ any constant for all portions of the beam i.e. $0 \le x \le 1$, then using $a_0 = a, n = 0, L = 1$ and all other constants zeros, we have from equations (25)-(28):

$$c_1 = 12\sum_{i=0}^{0+4}\frac{a_i L^{i+1}}{(i+1)(i+2)(i+3)(i+4)} - 6\sum_{i=0}^{0+3}\frac{a_i L^{i+1}}{(i+1)(i+2)(i+3)}$$

$$= \frac{12a_0L}{(0+1)(0+2)(0+3)(0+4)} - \frac{6a_0L}{(0+1)(0+2)(0+3)}$$

$$c_1 = \frac{12aL}{24} - \frac{6aL}{6} = \frac{aL}{2} - aL = -\frac{aL}{2}$$

$$c_2 = 2\sum_{i=0}^{0+3}\frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)} - 6\sum_{i=0}^{0+4}\frac{a_i L^{i+2}}{(i+1)(i+2)(i+3)(i+4)}$$

$$= \frac{2a_0L^2}{(0+1)(0+2)(0+3)} - \frac{6a_0L^2}{(0+1)(0+2)(0+3)(0+4)}$$

$$c_2 = \frac{2aL^2}{6} - \frac{6aL^2}{24} = aL^2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{aL^2}{12}$$

$$v(x) = \frac{1}{EI}\left\{\sum_{i=0}^{0+4}\frac{a_i x^{i+4}}{(i+1)(i+2)(i+3)(i+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right)\right\}$$

$$= \frac{1}{EI}\left\{\frac{a_0 x^{0+4}}{(0+1)(0+2)(0+3)(0+4)} + c_1\left(\frac{x^3}{6}\right) + c_2\left(\frac{x^2}{2}\right)\right\}$$

$$v(x) = \frac{1}{EI}\left\{\frac{ax^4}{24} + \left(-\frac{aL}{2}\right)\left(\frac{x^3}{6}\right) + \left(\frac{aL}{12}\right)\left(\frac{x^2}{2}\right)\right\}$$

$$= \frac{1}{EI}\left\{\frac{ax^4}{24} - \frac{aLx^3}{12} + \frac{aL^2x^2}{24}\right\}$$

$$v(x) = \frac{a}{24EI}\{x^4 - 2Lx^3 + L^2x^2\}$$

$$s(x) = \frac{1}{EI}\left\{\sum_{i=0}^{0+3}\frac{a_i x^{i+3}}{(i+1)(i+2)(i+3)}\left(\frac{x^2}{2}\right) + c_2x \atop +c_1\right\}$$

$$= \frac{1}{EI}\left\{\frac{a_0x^{0+3}}{(0+1)(0+2)(0+3)} \atop +\left(-\frac{aL}{2}\right)\left(\frac{x^2}{2}\right) + \left(\frac{aL^2}{12}\right)x\right\} \tag{29}$$

$$s(x) = \frac{1}{EI}\left\{\frac{ax^3}{6} - \frac{aLx^2}{4} + \frac{aL^2x}{12}\right\}$$

$$s(x) = \frac{a}{12EI}\{2x^3 - 3Lx^2 + L^2x] \tag{30}$$

For maximum deflection, $v' = 0$, and solving:

$$\frac{a}{24EI}\{4x^3 - 6Lx^2 + 2L^2x\} = 0$$
$$(or)\, 4x^3 - 6Lx^2 + 2L^2x = 0$$

It simplifies to:

$$x(2x - L)(x - L) = 0$$

So, critical points are:

$$x = 0, x = \frac{L}{2}, x = L$$

The double derivative test produces:

$$v'' = \frac{a}{24EI}\{12x^2 - 12Lx + 2L^2\}$$
$$v''(0) = \frac{a}{24EI}\{2L^2\} > 0, \quad mimima$$
$$v''\left(\frac{L}{2}\right) = \frac{a}{24EI}(3L^2 - 6L^2 + 2L^2)$$
$$= \frac{a}{24EI}(-L^2) < 0, \; maxima\ occurs\ at\ \frac{L}{2}$$
$$v''(L) = \frac{a}{24EI}(12L^2 - 12L^2 + 2L^2)$$
$$= \frac{a}{24EI}(2L^2) > 0, minima$$

Which is sufficient to show that the maximum deflection occurs at $x = \frac{L}{2}$, which is:

$$v_{max} = v\left(\frac{L}{2}\right) = \frac{a}{24EI}\left\{\left(\frac{L}{2}\right)^4 \atop -2L\left(\frac{L}{2}\right)^3 + L^2\left(\frac{L}{2}\right)^2\right\}$$
$$= \frac{a}{24EI}\left(\frac{L^4}{16} - \frac{L^4}{4} + \frac{L^4}{4}\right) \tag{31}$$
$$v_{max} = \frac{aL^4}{384EI}$$

Finally, from the equations (29)-(31), we conclude that these match exactly with those in the standard texts in Structural engineering and literature on elastic beams. This also deduces the validity of the proposed protocol used here for generalization and hence the results for the fixed TBM have been validated through the results on elastic beams.

CONCLUSION

In this study, the fixed TBM was considered and an attempt was made to get its general analytical solution which hold for all cases of practical load functions. For the purpose of generalization, a polynomial load function expressed as a Machlaurin's power series was considered. The successive integrations of the TBM with generalized load function lead us to derive equations of rotation and deflection for a fixed TBM. The claimed expressions lead to the particular expressions, and this was verified for four load functions: triangular, trapezoidal, parabolic and circular. Finally, sensitivity analysis was discussed for the rotation and deflection of a fixed TBM against the solely depend parameter in the non-dimensionalized form. For higher values of the parameter, the beam deflections were too higher to consider for stable regimen. The validation of the proposed protocol was successfully done through the differential equations model of elastic beams. The contributions of this study are useful for structural engineers and those practitioners who are working in the theory of beams and stability. The proposed general equations are computationally quicker than the traditional methods used to solve the model, and are also time efficient. These factors can be the reason of wider adaptability of the proposed protocol in future.

ACKNOWLEDGEMENT

The authors would like to thank the Mehran University of Engineering and Technology and Government College University, Pakistan for the support to conduct this research.

DECLARATION OF COMPETING INTEREST

None

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