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ANTIPLANE SHEAR MODE STRESS INTENSITY FACTOR FOR A SLIGHTLY PERTURBED CIRCULAR CRACK SUBJECT TO SHEAR LOAD

(Faktor Keamatan Regangan bagi Mod Ricih Anti-Satah untuk Retakan Bulat Sedikit Terusik Tertakluk kepada Beban Ricih)

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ABSTRACT

This paper deals with a slightly perturbed circular crack, Ω in the three dimensional plane. The problem of finding the resulting shear forces can be formulated as a hypersingular integral equation over a considered domain. Conformal mapping is used to transform the integral equation into a similar equation over a circular region, *D.* By making a suitable representation of hypersingular integral equation, the problem is reduced to solve a system of linear equations. The system is solved numerically for the unknown coefficients, which will later be used in determining the antiplane shear mode stress intensity factor. Comparison of the numerical solutions with the existing asymptotic solutions show a good agreement.

Keywords: hypersingular integral equation; conformal mapping; stress intensity factor

ABSTRAK

Dalam makalah ini dibincangkan retakan bulat sedikit terusik, Ω, dalam satah tiga dimensi. Masalah mencari tegasan ricih boleh diformulasikan sebagai persamaan kamiran hipersingular ke atas domain yang dipertimbangkan. Pemetaan menyebentuk dijanakan untuk mentransformasikan persamaan kamiran kepada suatu persamaan yang serupa ke atas kawasan bulat, *D*. Dengan membuat perwakilan persamaan kamiran hipersingular yang sesuai, masalah tersebut diturunkan kepada menyelesaikan sistem persamaan linear. Sistem ini diselesaikan secara berangka untuk menentukan pekali anu yang akan digunakan untuk menentukan faktor keamatan regangan bagi mod ricih anti-satah. Perbandingan keputusan berangka dengan penyelesaian asimptot yang sedia ada menunjukkan kesamaan.

Kata kunci: persamaan kamiran hipersingular; pemetaan menyebentuk; faktor keamatan regangan

1. Introduction

Fracture mechanics is an important tool for the analysis of cracked bodies and is introduced in order to analyse the relationship among stresses, cracks and fracture toughness which used in a wide range of industries. Different approaches have been used by many researches in finding the stress intensity factor along the crack edges and crack tips (Sneddon 1946; Theocaris & Ioakimides 1970; Weaver 1977; Tan 1983; Linkov & Mogilevskaya 1994; Zhu *et al.* 2001; Qin *et al*. 2008; Nik Long & Eshkuvatov 2009; Nik Long *et al*. 2011).

Potential method was adopted by Cruse (1969) in finding the solution of the unknown surface tractions and displacements in three dimensional electrostatic. Further, He (1973) applied the numerical procedure based on the boundary integral equation method in solving a penny shaped crack problem. The Somigliana formula is used by Guidera and Lardner (1975) to reduce an arbitrary elastic crack problem to a system of three integral equations for the components of displacement discontinuity. Moreover, the integral equations are solved explicitly for stresses and stress intensity factor for a penny shaped crack located in an infinite isotropic medium with the crack faces are subjected to arbitrary tractions. Mastrojannis *et al.* (1979) formulated the planar crack problem to the solution of a system of two dimensional Fredholm integral equation and solved numerically for determining the normal mode stress intensity factor. Cotterell and Rice (1980) used the perturbation method accurate to the first order in the derivation of stress intensity factors and his work can be extended to find the energy release rate at the tips of two dimensional kinked or slightly curved cracks. Similar approach can also be found in Rice (1985), Gao and Rice (1987) and Gao (1988). Another approach in solving a nearly circular crack can be found in Borodachev (1993). The almost circular cracks is solved by reducing the first kind Fredholm two dimensional integral equation of spacial problems of the theory of crack using the Fr´echet derivative of some nonlinear operator and variational formula. In particular, a solution of almost circular crack subjected to normal loading based on the inversion formula is obtained. Astiz (1986) derived a singular triangular element to compute the stress intensity factor along the crack border. Qin and Tang (1993) adopted the finite-part integral method in the derivation of a set of hypersingular integral equations and obtained the numerical results for the crack opening displacement and tensile mode stress intensity factor for the arbitrary flat crack in three dimensional elasticity. Nishimura *et al.* (1999) proposed a three-dimensional fast multipole boundary integral equation method for solving the crack problems. Particularly, the resulting numerical equation is solved using GMRES (generalised minimum residual method) in connection with FMM (fast multipole method). Wang (2001) obtained the exact solutions of stress intensity factors for the external circular crack problem in a three-dimensional infinite elastic body under asymmetric loadings using the boundary integral equation method. The two-dimensional singular boundary integral equations of the problem were reduced to a system of Abel integral equations by means of Fourier series and hypergeometric functions. Similar method also advocated by Theotokoglou (2004) when he solved the three dimensional planar cracks subjected to shear load. Whereas, Kiciak *et al.* (2003) determined the stress intensity factors for a variety of geometrical and stress field configurations subjected to complex stress fields based on the generalised weight functions. Recently, Aizikovich *et al.* (2010) employed dual integral equations in solving a penny-shaped tensile crack in an inhomogeneous elastic medium. It is showed that the approximate solution of the integral equation is asymptotically exact for both small and large values of the dimensionless geometric parameter of the problem.

In this present paper, we focus on the finding the numerical result for the antiplane shear mode stress intensity factor (mode 3) for a nearly circular crack via the solution of hypersingular integral equation and compare our computational result with Gao's (1988).

2. Problem Formulation

Consider a three dimensional infinite, homogenous, elastic and isotropic solid body containing a flat circular crack, Ω, located on the Cartesian coordinate (x, y, z) with origin O and Ω lies in the plane $z = 0$. Let the radious of the crack, Ω be a and

$$\Omega = \{(r, \theta): 0 \le r \le a, -\pi \le \theta < \pi\} .$$

Assume that the stress at infinity and the body force are absent. Now, equal and opposite $q_x(x, y)$ and $q_y(x, y)$ respectively, is applied to the crack plane, it is to be assumed that the z direction are traction free . Hence, in view of the shear load, the entire plane, must free from the normal stress, i.e.

$$\tau_{zz} = 0, \ \ z = 0.$$

The stress field can be found by considering the above problem subjected to the following mixed boundary condition on its surface $z = 0$:

$$\begin{aligned} \tau_{xz} &= \frac{\mu}{1-\nu} q_x(x,y) \in \Omega \\ \tau_{yz} &= \frac{\mu}{1-\nu} q_y(x,y) \in \Omega \\ u_x(x,y,z) &= u_y(x,y,z) = 0, (x,y) \in \partial\Omega \end{aligned} \quad (1)$$

where τ_{ij} is stress tensor, ν is the Possion's ratio , μ is the shear modulus, $\partial\Omega$ is the boundary of Ω and $q_x(x,y)$ and $q_y(x,y)$ are resultant forces in x and y directions, respectively. The problem satisfies the regularity conditions at infinity,

$$u_i(x,y,z) = O\left(\frac{1}{R}\right), \quad \tau_{ij}(x,y,z) = O\left(\frac{1}{R}\right), \quad i,j = x,y,z, \quad R \to \infty$$

where R is the distance,

$$R = \sqrt{(x-x_0)^2 + (y-y_0)^2}, \quad (x_0, y_0) \in \Omega$$

Rather to solve the mixed boundary value problem (1), it turns out to be more convenient to solve them separately, corresponding to concentrated force acting in x and y directions, i.e.

Problem 1 : $\tau_{xz} = \frac{\mu}{1-\nu} q_x(x,y)$, $\tau_{yz} = \tau_{zz} = 0, z = 0$ and

Problem 2 : $\tau_{yz} = \frac{\mu}{1-\nu} q_y(x,y)$, $\tau_{xz} = \tau_{zz} = 0, z = 0$

In this paper, we restrict the problem under consideration to the case where the stress on opposite crack surfaces are equal on x direction. Hence, the problem of finding the resulting force can be formulated as a hypersingular integral eq. (Nik Long *et al.* 2011)

$$\frac{1}{8\pi} H.S. \int_{\Omega} \frac{(2-\nu) + 3\nu e^{j2\Theta}}{R^3} w(x,y) d\Omega = q(x_0, y_0) \quad (2)$$

where $w(x,y)$ is the unknown crack opening dispalcement and the angle Θ is

$$x - x_0 = R cos\Theta \text{ and } y - y_0 = R sin\Theta.$$

The cross on the integral means the hypersingular, and it must be interpreted as a Hadamard finite part integral (Hadamard 2003) . Eq. (2) is to be solved subject to $w = 0$ on $\partial\Omega = 0$.

A polar coordinate (r, θ) is chosen so that the loading, $q(x,y)$ and crack opening displacement , $w(x,y)$ can be written as complex Fourier series

$$q(x,y) = \sum_{n=-\infty}^{n=\infty} q_n\left(\frac{r}{a}\right)e^{jn\theta}$$
$$w(x,y) = \sum_{n=-\infty}^{n=\infty} w_n\left(\frac{r}{a}\right)e^{jn\theta}.$$

Without lost of generality, we consider $a = 1$. Krenk (1979) showed that these formulas can be simplified if we expand q_n as a series expansions

$$q_n(r) = r^{|n|}\sum_{k=0}^{\infty} Q_k^n \frac{\Gamma\left(|n|+\frac{1}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}{(|n|+k)!\sqrt{1-r^2}} C_{2k+1}^{|n|+\frac{1}{2}}\left(\sqrt{1-r^2}\right)$$

$$w_n(r) = r^{|n|}\sum_{k=0}^{\infty} W_k^n \frac{\Gamma\left(|n|+\frac{1}{2}\right)k!}{\Gamma\left(|n|+k+\frac{3}{2}\right)} C_{2k+1}^{|n|+\frac{1}{2}}\left(\sqrt{1-r^2}\right)$$

where $j -$ complex expansion coefficients Q_k^n are assumed to be known, W_k^n are unknown and $C_m^\lambda(x)$ is the Gegenbaur polynomial of degree m and index λ (Erdélyi *et al.* 1953) . For a constant shear loading, $q(x,y) = -\tau$, the solution for a circular crack is obtainable.

3. Nearly Circular Crack

Let Ω be an arbitrary shaped crack of smooth boundary with respect to origin *O*, such that Ω is defined as

$$\Omega = \{(r,\theta): 0 \le r < \rho(\theta) < -\pi\}$$

where the boundary of Ω, $\partial\Omega$ is given by $r = \rho(\theta)$. Next, let the polar coordinate defined as $\zeta = se^{i\varphi}$ where $|\zeta| < 1$, hence, the unit disc *D* is given by

$$D = \{(s,\varphi): 0 \le s < 1, -\pi \le \varphi < \pi\}.$$

By the properties of Reimann Mapping theorem , a circular disc *D* is mapped conformally onto Ω using $z = af(\zeta)$. The analytic function *f* is known to exist for any simply connected domain Ω . In addition, we assume $|f'(\zeta)|$ is non zero and bounded for all $|\zeta| < 1$.

Consider the conformal mapping

$$f(\zeta) = \zeta + cg(\zeta) \tag{3}$$

where *c* is a dimensionless real parameter, *g* is an analytic function, with $g(\zeta) = \zeta^{m+1}$, m is an integer which maps the unit circular disc *D* in the ζ -plane into an arbitrary shape domain Ω in the *z*-plane,

$$\Omega = \{(r,\theta): 0 \le r < \rho(\theta) < -\pi\}$$

where $\partial\Omega$ is given by $r = \rho(\theta)$. This domain has a smooth, regular boundary for $0 \le (m+1)|c| \to 1$. As $(m+1)|c| \to 1$, one or more cusps develop. See Figs. 1 and 2 for $m = 1$ and $m = 2$, respectively, with various *c*.

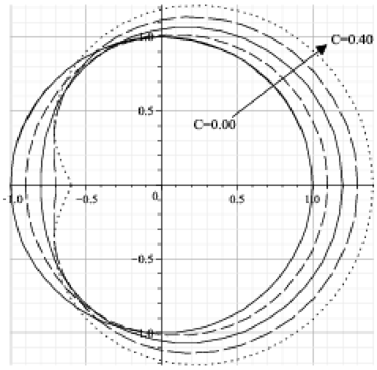

Figure 1: The Domain Ω For $m = 1$ With Different Choices of *c*

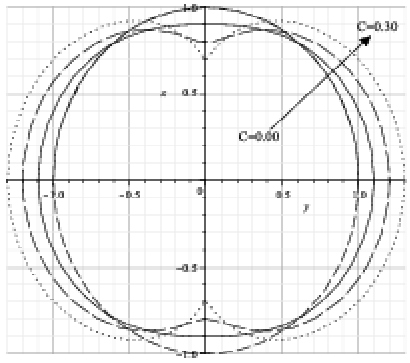

Figure 2: The Domain Ω For $m = 2$ With Different Choices of *c*

With the mapping (3), eq. (2) can be written as

$$\frac{2-v+3ve^{2j\Theta}}{8\pi} H.S\int_D \frac{W(\xi,\eta)}{S^3}d\xi d\eta + \frac{2-v}{8\pi}C.P\int_D W(\xi,\eta)K^{(1)}(\zeta,\zeta_0)d\xi d\eta + \frac{3v}{8\pi}\int_D W(\xi,\eta)K^{(2)}(\zeta,\zeta_0)d\xi d\eta = Q(\xi_0,\eta_0) \quad (4)$$

where the $K^{(1)}(\zeta,\zeta_0)$ and $K^{(2)}(\zeta,\zeta_0)$ are

$$K^{(1)}(\zeta,\zeta_0) = \frac{|f'(\zeta)|^{-\frac{3}{2}}|f'(\zeta_0)|^{-\frac{3}{2}}}{|f(\zeta)-f(\zeta_0)|^3}e^{j(\delta-\delta_0)} - \frac{1}{|\zeta-\zeta_0|^3}$$

$$K^{(2)}(\zeta,\zeta_0) = \frac{|f'(\zeta)|^{-\frac{3}{2}}|f'(\zeta_0)|^{-\frac{3}{2}}}{|f(\zeta)-f(\zeta_0)|^3}e^{j(2\Theta-\delta-\delta_0)} - \frac{1}{|\zeta-\zeta_0|^3}e^{2j\Phi}.$$

This hypersingular integral equation over a circular disc *D* is to be solved subject to $W = 0$ on $s = 1$. For small *S*, $K^{(1)}$ is Cauchy-type singular kernel while $K^{(2)}$ is weakly singular with $O(S^{-1})$ (Nik Long *et al.* 2011).

Denote W as

$$W(\xi,\eta) = \sum_{n,k} W_k^n A_k^n(s,\varphi) \quad (5)$$

where W_k^n are unknown coeefcients and $A_k^n(s,\varphi)$ is defined by

$$A_k^n(s,\varphi) = s^{|n|}C_{2k+1}^{|n|+\frac{1}{2}}\left(\sqrt{1-s^2}\right)e^{jn\varphi}\,. \quad (6)$$

Introduce

$$L_h^m(s,\varphi) = s^{|m|} C_{2h+1}^{|m|+\frac{1}{2}}\left(\sqrt{1-s^2}\right)\cos(m\varphi). \tag{7}$$

The relationship of these two functions, $A_k^n(s,\varphi)$ and $L_h^m(s,\varphi)$ can be expressed as

$$\int_\Omega \frac{Y_k^n(\zeta)L_h^m(\zeta)sdsd\varphi}{\sqrt{1-s^2}} = B_k^n \delta_{hk}\delta_{nm} \tag{8}$$

where δ_{ij} is the Kroneker delta and

$$B_k^n = \frac{\sigma_n}{2} h_{2k+1}^{n+\frac{1}{2}},$$

$$h_{2k+1}^{n+\frac{1}{2}} = \frac{\pi}{2^{2n}} \frac{\Gamma(2k+2n+2)}{\left(2k+n+\frac{3}{2}\right)(2k+1)!\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2},$$

$$\sigma_n = \begin{cases} 2\pi & n=0 \\ \pi & n \neq 0 \end{cases}.$$

Substitute Eq. (5) into (4) and making use of Krenk's formula (1979) yields

$$\sum_{n,k} F_k^n(s_0,\varphi_0)\, W_k^n = Q\big(\xi_0(s_0,\varphi_0), \eta_0(s_0,\varphi_0)\big) \tag{9}$$

where

$$\begin{aligned} F_k^n(s_0,\varphi_0) \quad &= -E_k^n \frac{\left(2-\nu+3\nu e^{2j\Theta}\right)A_k^n(s_0,\varphi_0)}{2\sqrt{1-s_0^2}} + \frac{2-\nu}{8\pi}\int_D A_k^n(s,\varphi)K^{(1)}(\zeta,\zeta_0)d\xi d\eta \\ &+ \frac{3\nu}{8\pi}\int_D A_k^n(s,\varphi)K^{(2)}(\zeta,\zeta_0)d\xi d\eta\, ; 0 < s \leq 1, 0 \leq \varphi < 1 \end{aligned}$$

and $\sum_{n,k} = \sum_{n=-N_1}^{N_1}\sum_{k=0}^{N_2}$.

Apply the Galerkin method in evaluating the Eq. (9) leads to

$$\begin{aligned} &\sum_{n,k} \breve{W}_k^n \left(-\frac{2-\nu+3\nu e^{2k\Theta}}{2}\delta_{hk}\delta_{|n||m|} + S_{hk}^{mn}\right) = Q_h^m\, ; \\ &-N_1 \leq m \leq N_1\, ; 0 \leq k \leq N_2 \end{aligned} \tag{10}$$

where

$$S_{hk}^{mn} = \frac{2-\nu}{8\pi\sqrt{E_k^m B_k^m}\sqrt{E_h^n B_h^n}} T_{hk}^{mn}, \qquad T_{hk}^{mn} = \int_D L_h^m(\zeta) \int_D Y_k^n(\zeta) H(\zeta,\zeta_0) d\zeta d\zeta_0,$$

$$Q_k^n = \frac{1}{\sqrt{E_k^n B_k^n}} \int_D Y_k^n(\zeta_0) Q(\zeta_0) d\zeta_0, \qquad W_k^n = -\breve{W}_k^n \, G_{2k+1}^{|n|+1/2} \sqrt{\frac{E_k^n}{B_k^n}}$$

and

$$H(\zeta,\zeta_0) = (2-\nu)K^{(1)}(\zeta,\zeta_0) + 3\nu K^{(2)}(\zeta,\zeta_0).$$

In (10) , we have used the following notation :

$\zeta_0 = \zeta_0(s_0,\varphi_0)$, $Q(\zeta_0) = Q(s_0 cos\varphi_0, s_0 \sin\varphi_0)$ and $d\zeta_0 = s_0 dsd\varphi$.

In evaluating the quadruple integral in Eq. (10), we have used the Gaussian quadrature and trapezoidal formulas for the radial and angular directions, with the appropriate choice of collocation points (s,φ) and (s_0,φ_0) , respectively.

4. Antiplane Stress Intensity Factor

The important parameter in the crack problem is to determine the stress intensity factor. The antiplane shear mode stress intensity factor, $K_3(\varphi)$ is defined as (Gao & Rice 1986, 1987) and (Gao 1988)

$$K_3(\varphi) = \lim_{r\to a} M_3 \sqrt{\frac{2\pi}{a-r}} w(x,y) \tag{11}$$

where M_3 are constants. Let $a(\varphi) = |f(e^{i\varphi})|, r = |f(se^{i\varphi})|$, and as *s* close to 1, we have

$$|f(e^{i\varphi}) - f(se^{i\varphi})| = (1-\mathrm{s})|f'(e^{i\varphi})|. \tag{12}$$

Substitute Eq. (5) and (12) into (13) gives

$$K_3(\varphi) = M_3 \left\{ |f'(e^{i\varphi})|^{-1} \sum_{n,k} \frac{\breve{W}_k^n}{\sqrt{E_k^n B_k^n}} Y_k^n(\varphi) \right\} \tag{13}$$

where

$Y_k^n(\varphi) = D_{2k+1}^{|n|+\frac{1}{2}}(0)\cos(n\varphi)$, $M_3 = \frac{1-\nu}{8\pi}\sin(\varphi)$ and

$C_{2k+1}^{|n|+\frac{1}{2}}(\sqrt{1-s^2}) = \sqrt{1-s^2} D_{2k+1}^{|n|+\frac{1}{2}}(\sqrt{1-s^2})$.

5. Result and Discussion

Tables 1 and 2 show that our numerical scheme converges rapidly with a small value of $N = N_1 = N_2$.

Table 1: Numerical convergence antiplane mode stress intensity factor for $f(\zeta) = \zeta + 0.1\zeta^2$

N	$K_3(0.00)$	$K_3\left(\frac{\pi}{4}\right)$	$K_3\left(\frac{\pi}{2}\right)$	$K_3\left(\frac{3\pi}{4}\right)$	$K_3(\pi)$
0	0.0000	-6.330E-04	-9.5075E-04	-7.2780E-04	-1.3145E-19
1	0.0000	-0.7174	-0.9211	-0.5854	-9.6785E-17
2	0.0000	-0.7174	-0.9199	-0.5854	-9.6785E-17
3	0.0000	-0.7174	-0.9199	-0.5854	-9.6785E-17
4	0.0000		-0.9199	-0.5854	-9.6785E-17
5				-0.5854	-9.6785E-17
6				-0.5854	

Table 2: Numerical convergence antiplane mode stress intensity factor for $f(\zeta) = \zeta + 0.45\zeta^2$

N	$K_3(0.00)$	$K_3\left(\frac{\pi}{4}\right)$	$K_3\left(\frac{\pi}{2}\right)$	$K_3\left(\frac{3\pi}{4}\right)$	$K_3(\pi)$
0	0.0000	-3.6543E-04	-5.9039E-04	-5.6560E-04	-2.6519E-19
1	0.0000	0.0000	-4.3808E-03	-7.3368E-03	-3.4300E-18
2	0.0000	-0.9204	-0.8734	-0.2892	-3.4803E-17
3	0.0000	-0.9179	-0.8734	-0.2930	-3.2324E-17
4	0.0000	-0.9102	-0.8715	-0.2948	-3.1443E-17
5		-0.9189	-0.8715	-0.2954	-3.0878E-17
6		-0.9188	-0.8719	-0.2954	-3.0785E-17
7		-0.9188	-0.8718	-0.2952	-3.0708E-17
8		-0.9188	-0.8718	-0.2952	-3.0693E-17
9			-0.8718	-0.2952	-3.0685E-17
10				-0.2952	-3.0681E-17
11				-0.2952	-3.0679E-17
12					-3.0679E-17
13					-3.0679E-17
14					-3.0679E-17

Next, we compare our result for the determination of the antiplane shear mode stress intensity factor, (Eq. (13)) with the asymptotic solutions obtained by Gao (1988) These are shown in Figs. 3 and 4 for *c*=0.1 at $m = 1$ and $m = 2$, respectively. Whereas Fig. 5 displayed the comparison result for *c*=-0.2 at $m = 1$. Our numerical results seems to agree with the asymptotic solution obtained by Gao (1988).

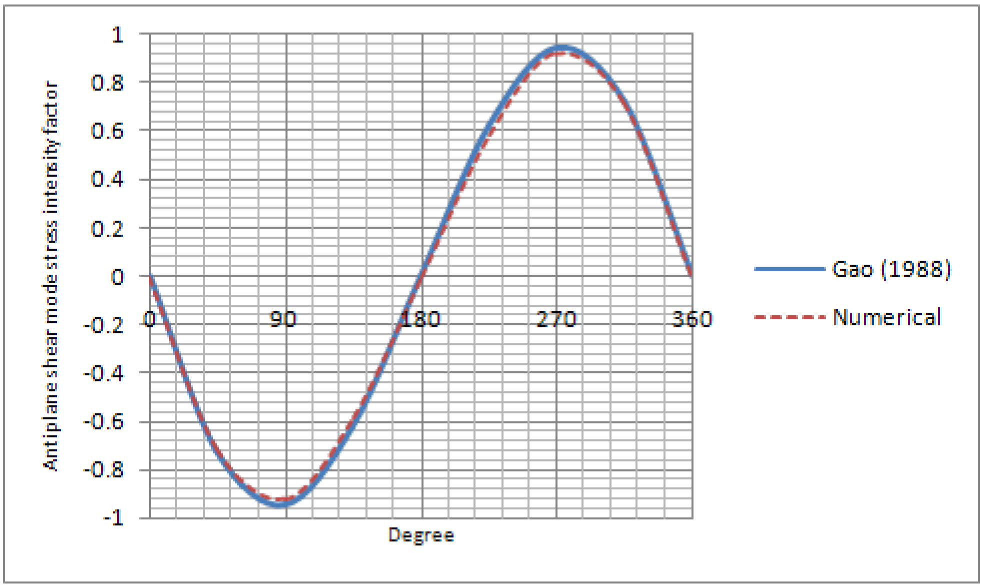

Figure 3: The stress intensity factor $K_3(\varphi)$ as function $f(\zeta) = \zeta + c\zeta^2$ when $c = 0.1$

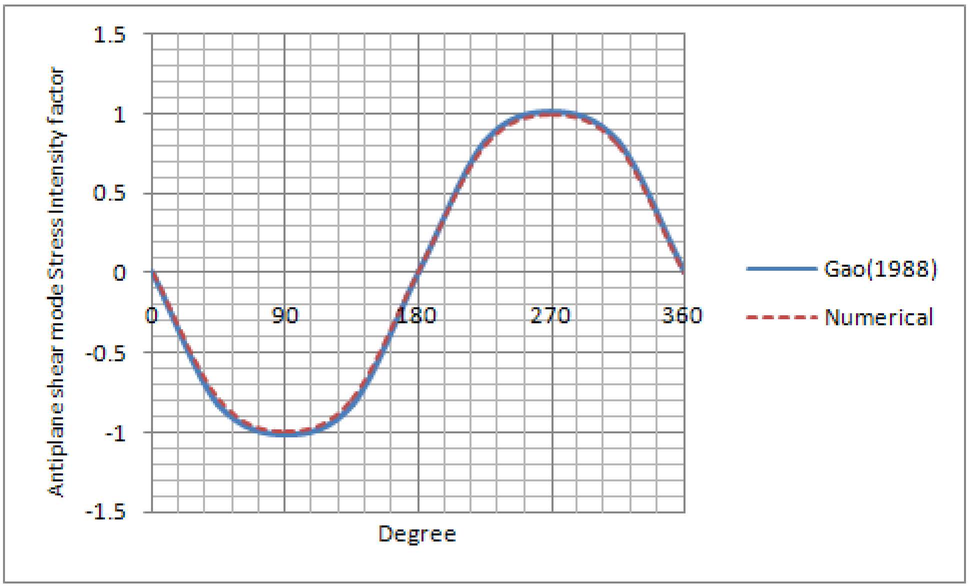

Figure 4: The stress intensity factor $K_3(\varphi)$ as function $f(\zeta) = \zeta + c\zeta^3$ when $c = 0.1$

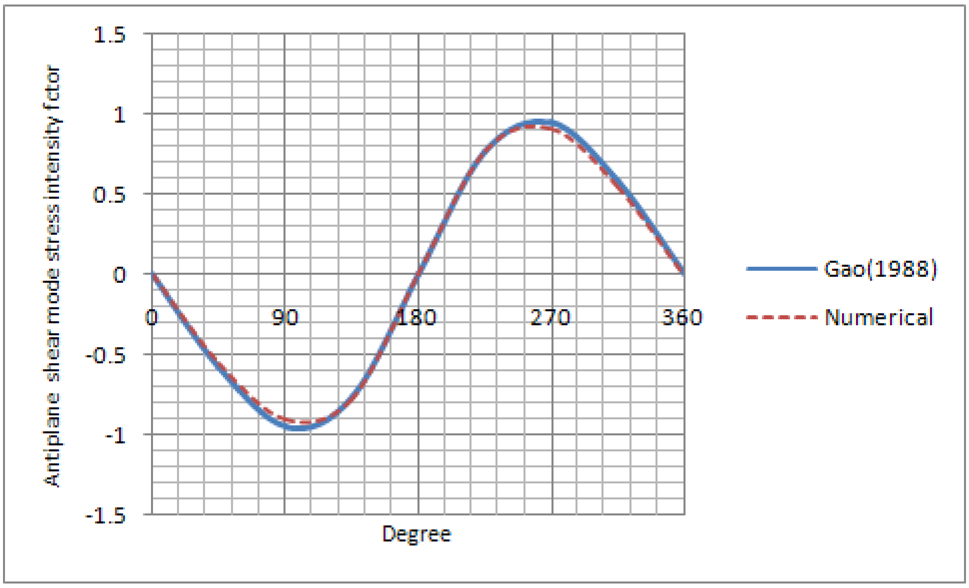

Figure 4: The stress intensity factor $K_3(\varphi)$ as function $f(\zeta) = \zeta + c\zeta^2$ when $c = -0.2$

6. Conclusion

In the present paper, the nearly circular crack is mapped conformally into a unit circle. Through this mapping, the equation is transformed into hypersingular integral equation over a circular crack, which enable us to use the formula obtained by Krenk (1979). By choosing the appropriate collocation points, this equation is reduced into a system of linear equations and solved for the unknown coefficients, which are later used in finding the antiplane shear mode stress intensity factor. Through a careful analysis and comparison between the present solutions and Gao (1988), it was shown that our numerical results agree with the existing asymptotic solution.

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