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CERTAIN PROPERTIES FOR ANALYTIC FUNCTIONS DEFINED BY A GENERALISED DERIVATIVE OPERATOR

(Sifat Tertentu bagi Fungsi Analisis yang Ditakrif oleh Pengoperasi Terbitan Teritlak)

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ABSTRACT

In this paper, some important properties of analytic functions with negative coefficients defined by a generalised derivative operator are investigated. The properties include the necessary and sufficient conditions, radius of starlikeness, convexity and close-to-convexity.

Keywords: derivative operator; radius of starlikeness; convexity; close-to-convexity

ABSTRAK

Dalam makalah ini dikaji beberapa sifat penting bagi fungsi analisis berpekali negatif yang ditakrif oleh pengoperasi terbitan teritlak. Sifat tersebut termasuklah syarat perlu dan cukup, jejari kebakbintangan, kecembungan dan dekat-dengan-kecembungan.

Kata kunci: pengoperasi terbitan; jejari kebakbintangan; kecembungan; dekat-dengan-kecembungan

1. Introduction

Let A denote the class of functions f in the open unit disc

$$\mathrm{U} = \{z \in \mathrm{C} : |z| < 1\},$$

and let T denote the subclass of A consisting of analytic functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathrm{U}),$$

which are analytic in the unit disc U.

Definition 1.1 *Let $f \in A$. Then f is said to be convex of order $\mu\ (0 \le \mu < 1)$ if and only if*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \mu, z \in \mathrm{U}.$$

Definition 1.2 *Let $f \in A$. Then f is said to be starlike of order $\mu\ (0 \le \mu < 1)$ if and only if*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \mu, z \in \mathrm{U}.$$

Amer and Darus (2011; 2012) have recently introduced a new generalised derivative operator $I^m(\lambda_1, \lambda_2, l, n) f(z)$ as follows:

Definition 1.3 *Let* $f \in A$, *then the generalised derivative operator is given by*

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = z - \sum_{k=2}^{\infty}\zeta_k a_k z^k, \tag{1}$$

where

$$\zeta_k = \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k), \quad n,m \in N_0 = \{0,1,2,...\},$$

$$\lambda_2 \geq \lambda_1 \geq 0, l \geq 0, \text{ and } c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}.$$

Definition 1.4 *Let a function* f *be in T. Then* f *is said to be in the class of* $\tau(\alpha,\beta,\lambda_1,\lambda_2,l,n)$ *if,*

$$\Re\left\{\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z[I^m(\lambda_1,\lambda_2,l,n)f(z)]'}\right\} > \alpha\left|\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z[I^m(\lambda_1,\lambda_2,l,n)f(z)]'}-1\right|+\beta, \tag{2}$$

where $n,m \in N_0 = \{0,1,2,...\}, l \geq 0,\ 0 \leq \alpha < 1$, and $0 \leq \beta < 1$.

The family $\tau(\alpha,\beta,\lambda_1,\lambda_2,l,n)$ is a special interest as it contains many well-known classes of analytic univalent functions. This family is studied by (Najafzadeh & Ebadian 2009), and also (Tehranchi & Kulkarni 2006a; 2006b).

2. Necessary and Sufficient Conditions

Theorem 2.1 *Let* $f \in T$. *Then* $f \in \tau(\alpha,\beta,\lambda_1,\lambda_2,l,n)$ *if and only if,*

$$\sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}\zeta_k a_k < 1. \tag{3}$$

Proof: Let us assume that $f \in \tau(\alpha,\beta,\lambda_1,\lambda_2,l,n)$. So by using the fact that $\Re(\omega) > \alpha\,|\,\omega - 1\,| + \beta$ *if, and only if* $\Re[\omega(1+\alpha e^{i\theta}) - \alpha^{i\theta}] > \beta$

and letting $\omega = \dfrac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z[I^m(\lambda_1,\lambda_2,l,n)f(z)]'}$ in (2), we obtain $\Re[\omega(1+\alpha e^{i\theta}) - \alpha^{i\theta}] > \beta.$

So

$$\Re\left[\frac{z-\sum_{k=2}^{\infty}\zeta_k a_k z^k}{z\left(1-\sum_{k=2}^{\infty}k\,\zeta_k a_k z^{k-1}\right)}(1+\alpha e^{i\theta})-\alpha e^{i\theta}-\beta\right]>0,$$

then

$$\Re\left[\frac{1-\beta-\sum_{k=2}^{\infty}(1-\beta k)\zeta_k a_k z^{k-1}-\alpha e^{i\theta}\sum_{k=2}^{\infty}(1-k)\zeta_k a_k z^{k-1}}{1-\sum_{k=2}^{\infty}k\,\zeta_k a_k z^{k-1}}\right]>0.$$

The above inequality must hold for all z in U. Letting $z=re^{-i\theta}$ where $0\le r<1$, we obtain

$$\Re\left[\frac{1-\beta-\sum_{k=2}^{\infty}[(1-\beta k)+\alpha e^{i\theta}(1-k)]\zeta_k a_k r^{k-1}}{1-\sum_{k=2}^{\infty}k\,\zeta_k a_k r^{k-1}}\right]>0.$$

By letting $r\to 1$ through half line $z=re^{-i\theta}$ and by mean value theorem, we have

$$\Re\left[1-\beta-\sum_{k=2}^{\infty}[(1-\beta k)+\alpha(1-k)]\zeta_k a_k r^{k-1}\right]>0,$$

then we get

$$\sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}\zeta_k a_k<1.$$

Conversely, let (3) holds. We will show that (2) is satisfied and so $f\in\tau(\alpha,\beta,\lambda_1,\lambda_2,l,n)$. By using the fact that $\Re(\omega)>\beta$ *if and only if* $|\omega-(1+\beta)|<|\omega+(1-\beta)|$, it is enough to show that

$$|\omega-(1+\alpha|\omega-1|+\beta)|<|\omega+(1-\alpha|\omega-1|+\beta)|,\ \text{if } R=|\omega+(1-\alpha|\omega-1|+\beta)|$$

$$=\frac{1}{|z[I^m(\lambda_1,\lambda_2,l,n)f(z)]'|}\left|2z-\beta z-\sum_{k=2}^{\infty}[1+(1-\beta)+\alpha-\alpha k]\zeta_k a_k z^k\right|.$$

This implies that

$$R>\frac{|z|}{|z[I^m(\lambda_1,\lambda_2,l,n)f(z)]'|}\left|2-\beta-\sum_{k=2}^{\infty}[k+(1+\alpha)-k(\alpha+\beta)k]\zeta_k a_k z^k\right|.$$

Similarly, if $L = |\omega - (1-\alpha|\omega-1|+\beta)|$, we get

$$L < \frac{|z|}{|z[I^m(\lambda_1,\lambda_2,l,n)f(z)]'|}\left|\beta + \sum_{k=2}^{\infty}[-k+(1+\alpha)-k(\alpha+\beta)k]\zeta_k a_k z^k\right|.$$

It is easy to verify that $R-L>0$ and so the proof is complete.

Corollary 2.1 *Let* $f \in \tau(\alpha,\beta,\lambda_1,\lambda_2,l,n)$, *then*

$$a_k < \frac{1-\beta}{[(1+\alpha)-k(\alpha+\beta)]\zeta_k}.$$

Proof: For $0 \le \mu < 1$, we need to show that $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\mu.$
Now, let us show that

$$\left|\frac{zf'(z)-f(z)}{f(z)}\right| = \left|\frac{-\sum_{k=2}^{\infty}(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}a_k z^{k-1}}\right| \le \frac{\sum_{k=2}^{\infty}(k-1)a_k |z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k |z|^{k-1}} < 1-\mu$$

$$\Rightarrow \sum_{k=2}^{\infty}a_k |z|^{k-1}\left(\frac{k-\mu}{1-\mu}\right) < 1.$$

By Theorem 2.1, it is enough to consider

$$|z|^{k-1} < \frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{(k-\mu)(1-\beta)}\zeta_k.$$

□

Theorem 2.2 *Let* $f \in \tau(\alpha,\beta,\lambda_1,\lambda_2,l,n)$. *Then* f *is convex of order* $\mu (0 \le \mu < 1)$ *in* $|z| < r = r_2(\alpha,\beta,\lambda_1,\lambda_2,l,n,\mu)$ *where*

$$r_2(\alpha,\beta,\lambda_1,\lambda_2,l,n,\mu) = \inf_k\left[\frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{k(k-\mu)(1-\beta)}\zeta_k\right]^{\frac{1}{k-1}}.$$

Proof: For $0 \le \mu < 1$, we need to show that $\left|\frac{zf''(z)}{f'(z)}\right| < 1-\mu.$

Now again let us show that

$$\left|\frac{-\sum_{k=2}^{\infty} k(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty} k a_k z^{k-1}}\right| \le \frac{\sum_{k=2}^{\infty} k(k-1)a_k |z|^{k-1}}{1-\sum_{k=2}^{\infty} k a_k |z|^{k-1}} < 1-\mu$$

$$\Rightarrow \sum_{k=2}^{\infty} k a_k |z|^{k-1}\left(\frac{k-\mu}{1-\mu}\right) < 1.$$

By Theorem 2.1, it is again enough to consider

$$|z|^{k-1} < \frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{k(k-\mu)(1-\beta)}\zeta_k.$$

□

Theorem 2.3 *Let* $f(z) \in \tau(\alpha,\beta,\lambda_1,\lambda_2,l,n)$. *Then* $f(z)$ *is close-to-convex of order* $\mu(0 \le \mu < 1)$ *in* $|z| < r = r_3(\alpha,\beta,\lambda_1,\lambda_2,l,n,\mu)$ *where*

$$r_3(\alpha,\beta,\lambda_1,\lambda_2,l,n,\mu) = \inf_k \left[\frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{k(1-\beta)}\zeta_k\right]^{\frac{1}{k-1}}.$$

Proof: For $0 \le \mu < 1$, we must show that $|f'(z)-1| < 1-\mu$.
Similarly we show that

$$|f'(z)-1| = \left|\sum_{k=2}^{\infty} k a_k z^{k-1}\right| \le \sum_{k=2}^{\infty} k a_k |z|^{k-1} \le 1-\mu$$

$$\Rightarrow \sum_{k=2}^{\infty} \frac{k}{1-\mu} a_k |z|^{k-1} < 1.$$

By Theorem 2.1, the above inequality holds true if,

$$|z|^{k-1} < \frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{k(1-\beta)}\zeta_k.$$

□

3. Radius of Starlikeness, Convexity and Close-to-convexity

In this section, we will calculate *Radius of Starlikeness, Convexity and Close-to-convexity* for the class $\tau(\alpha,\beta,\lambda_1,\lambda_2,l,n)$.

Theorem 3.1 *Let* $f \in \tau(\alpha,\beta,\lambda_1,\lambda_2,l,n)$. *Then* f *is starlike of order* $\mu(0 \le \mu < 1)$ *in* $|z| < r = r_1(\alpha,\beta,\lambda_1,\lambda_2,l,n,\mu)$, *where*

$$r_1(\alpha,\beta,\lambda_1,\lambda_2,l,n,\mu) = \inf_k \left[\frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{(k-\mu)(1-\beta)} \zeta_k \right]^{\frac{1}{k-1}}.$$

Proof: For $0 \le \mu < 1$, we need to show that $\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1-\mu.$
Now, we have to show that

$$\left| \frac{zf'(z)-f(z)}{f(z)} \right| = \left| \frac{-\sum_{k=2}^{\infty}(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}a_k z^{k-1}} \right| \le \frac{\sum_{k=2}^{\infty}(k-1)a_k |z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k |z|^{k-1}} < 1-\mu$$

$$\Rightarrow \sum_{k=2}^{\infty} a_k |z|^{k-1} \left(\frac{k-\mu}{1-\mu} \right) < 1.$$

By Theorem 2.1, it is enough to consider

$$|z|^{k-1} < \frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{(k-\mu)(1-\beta)} \zeta_k.$$

□

Theorem 3.2 *Let* $f \in \tau(\alpha,\beta,\lambda_1,\lambda_2,l,n)$. *Then* f *is convex of order* $\mu(0 \le \mu < 1)$ *in* $|z| < r = r_2(\alpha,\beta,\lambda_1,\lambda_2,l,n,\mu)$, *where*

$$r_2(\alpha,\beta,\lambda_1,\lambda_2,l,n,\mu) = \inf_k \left[\frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{k(k-\mu)(1-\beta)} \zeta_k \right]^{\frac{1}{k-1}}.$$

Proof: For $0 \le \mu < 1$, we need to show that $\left| \frac{zf''(z)}{f'(z)} \right| < 1-\mu.$

We have to show that

$$\left|\frac{-\sum_{k=2}^{\infty} k(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty} ka_k z^{k-1}}\right| \le \frac{\sum_{k=2}^{\infty} k(k-1)a_k |z|^{k-1}}{1-\sum_{k=2}^{\infty} ka_k |z|^{k-1}} < 1-\mu$$

$$\Rightarrow \sum_{k=2}^{\infty} ka_k |z|^{k-1}\left(\frac{k-\mu}{1-\mu}\right) < 1.$$

By Theorem 2.1, it is enough to consider

$$|z|^{k-1} < \frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{k(k-\mu)(1-\beta)}\zeta_k .$$

Theorem 3.3 *Let* $f \in \tau(\alpha,\beta,\lambda_1,\lambda_2,l,n)$. *Then* f *is close-to-convex of order* $\mu(0\le\mu<1)$ *in* $|z|< r = r_3(\alpha,\beta,\lambda_1,\lambda_2,l,n,\mu)$, *where*

$$r_3(\alpha,\beta,\lambda_1,\lambda_2,l,n,\mu) = \inf_k\left[\frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{k(1-\beta)}\zeta_k\right]^{\frac{1}{k-1}}.$$

Proof: For $0\le\mu<1$, we must show that $|f'(z)-1|<1-\mu.$

We have to show that

$$|f'(z)-1| = \left|\sum_{k=2}^{\infty} ka_k z^{k-1}\right| \le \sum_{k=2}^{\infty} ka_k |z|^{k-1} \le 1-\mu$$

$$\Rightarrow \sum_{k=2}^{\infty}\frac{k}{1-\mu}a_k |z|^{k-1} < 1.$$

By Theorem 2.1, the above inequality holds true if

$$|z|^{k-1} < \frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{k(1-\beta)}\zeta_k .$$

□

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