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Instability to Nonlinear Vector Differential Equations of Fourth Order with Constant Delay

(Ketakstabilan Persamaan Pembeza Vektor Tak Linear Keempat dengan Tundaan Malar)

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ABSTRACT

We consider a vector nonlinear differential equation of fourth order with a constant delay. We establish new sufficient conditions, which guarantee the instability of the zero solution of that equation. An example is given to illustrate the theoretical analysis made in this paper.

Keywords: *Delay*; *fourth order*; *instability*; *vector differential equation*

ABSTRAK

Kami telah pertimbangkan persamaan pembeza vektor taklinear tertib keempat dengan tundaan malar. Kami tunjukkan keadaan mencukupi yang baru yang menjamin ketakstabilan penyelesaian sifar persamaan tersebut. Satu contoh diberikan untuk menunjukkan analisis teori yang dilakukan dalam kertas ini.

Kata kunci: Kestabilan; tundaan; persamaan pembeza vektor; tertib keempat

INTRODUCTION

Sun and Hou (1999) considered the scalar nonlinear differential equation of the fourth order:

$$x^{(4)} + a_1\dddot{x} + h(\dot{x})\ddot{x} + g(x)\dot{x} + f(x) = 0, \tag{1}$$

where a_1 is a constant. The authors established some sufficient conditions, which guarantee the instability of the solution $x = 0$ of (1).

Later, Tunç (2011c) considered (1) in its vector form as follows:

$$X^{(4)} + A\dddot{X} + H(\dot{X})\ddot{X} + G(X)\dot{X} + F(X) = 0, \tag{2}$$

The author presented a result on the instability of the zero solution of (2).

In this paper, instead of (2), we considered its delay form as follows:

$$X^{(4)} + A\dddot{X} + H(\dot{X})\ddot{X} + G(X)\dot{X} + F(X(t-\tau)) = 0, \tag{3}$$

where $X \in \Re^n$, $\tau > 0$ is the constant deviating argument, A is a constant $n \times n$-symmetric matrix, H and G are continuous $n \times n$-symmetric matrix functions for the arguments displayed explicitly, $F: \Re^n \to \Re^n$, $F(0) = 0$ and F is continuous for all $X \in \Re^n$. It is assumed the existence and the uniqueness of the solutions of (1).

Equation (3) is the vector version for systems of real fourth order nonlinear differential equations of the form:

$$\begin{aligned} &x_i^{(4)} + \sum_{k=1}^{n} a_{ik}x_k''' + \sum_{k=1}^{n} h_{ik}\left(x_1', x_2', \ldots, x_n'\right)x_k'' \\ &+\sum_{k=1}^{n} g_{ik}\left(x_1, x_2, \ldots, x_n\right)x_k' \\ &+f_i(x_1(t-\tau), x_2(t-\tau), \ldots, x_n(t-\tau)) = 0, \\ &(i = 1, 2, \ldots, n). \end{aligned}$$

Instead of (3), we considered its equivalent differential system:

$$\dot{X} = Y,\ \dot{Y} = Z,\ \dot{Z} = W,$$

$$\dot{W} = -AW - H(Y)Z - G(X)Y - F(X) + \int_{t-\tau}^{t} J_F(X(s))Y(s)ds, \tag{4}$$

which was obtained by setting $\dot{X} = Y$, $\ddot{X} = Z$, $\dddot{X} = W$ from (3).

Let $J_F(X)$ and $J(H(T)Y|Y)$ denote the linear operators from $F(X)$ and $H(Y)$ to

$$J_F(X) = \left(\frac{\partial f_i}{\partial x_j}\right),$$

and

$$\begin{aligned} J\left(H(Y)Y|Y\right) &= \left(\frac{\partial}{\partial y_j}\sum_{k=1}^{n} h_{ik}y_k\right) \\ &= H(Y) + \left(\sum \frac{\partial h_{ik}}{\partial y_j}y_k\right), (i, j = 1, 2, \ldots, n), \end{aligned}$$

where $(x_1,\ldots, x_n)$, $(y_1,\ldots,y_n)$, $(f_1,\ldots, f_n)$ and (h_{ik}) are components of X, Y, F and H, respectively. In what follows, it was assumed that $J_F(X)$ and exist and are symmetric and continuous.

Meanwhile, it should be noted that in the past few decades, the instability of the solutions of various scalar differential equations of fourth order without and with delay and vector differential equations of fourth order without delay was discussed in the literature. For a comprehensive treatment of the subject, we refer the readers to the papers of Ezeilo (1978, 1979, 2000); Sadek (2003); Tunç (2004, 2006, 2009, 2010, 2011a, 2011b, 2011c) and the references cited in these sources. However, to the best of our knowledge from the literature, the instability of solutions for the vector differential equations of the fourth order with a deviating argument has not been discussed in the literature. This paper is the first attempt and work on the topic for the vector differential equations of fourth order with a deviating argument. The motivation to produce this paper comes from the above papers done on scalar differential without and with delay and vector differential equations without delay. Our aim was to achieve the results established in Sun and Hou (1999) and Tunç (2011c) to (3) with a deviating argument. By this work, we improved the results of Sun and Hou (1999) and Tunç (2011c) to a vector differential equation of fourth order with delay. Based on Krasovskii's criterions (Krasovskii 1955), we proved our main result and an example is also provided to illustrate the feasibility of the proposed result. The result is new and different from that in the papers mentioned above.

The symbol $\langle X,Y\rangle$ correspondings to any pair X, Y in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, that is, $\langle X,Y\rangle = \sum_{i=1}^{n} x_i y_i$; thus $\langle X,Y\rangle = \|X\|^2$, and $\lambda_i(\Omega)$, $(i = 1,2,\ldots,n)$, are the eigenvalues of the real symmetric $n \times n$- matrix Ω. The matrix Ω is said to be negative-definite, when $\langle \Omega X, Y\rangle \le 0$ for all nonzero X in $\Re^n$.

Main Result

Before the introduction of the main result, we need the following results.

Lemma 1 (Bellman 1997). *Let A be a real symmetric n × n-matrix and*

$$a' \ge \lambda_i(A) \ge a > 0,\ (i = 1,2,\ldots,n),$$

where a' and a are constants.

Then $a'\langle X,X\rangle \ge \langle AX,X\rangle \ge a^2\langle X,X\rangle$, and

$$a'^2\langle X,X\rangle \ge \langle AX,X\rangle \ge a^2\langle X,X\rangle.$$

In the following theorem, we gave a basic idea of the method about the instability of solutions of ordinary differential equations. The following theorem is due to Četaev's, (LaSalle & Lefschetz 1961).

Theorem 1 (Instability Theorem of Četaev's). Let Ω be a neighborhood of the origin. Let there be given a function $V(x)$ and region Ω_1 in Ω with the following properties:

(i) $V(x)$ has continuous first partial derivatives in Ω_1.
(ii) $V(x)$ and $\dot{V}(x)$ are positive in Ω_1.
(iii) At the boundary points of Ω_1 inside Ω, $V(x) = 0$.
(iv) The origin is a boundary point of Ω_1.

Under these conditions the origin is unstable.

Let $r \ge 0$ be given and let $C = C([-r,0], \Re^n)$ with

$$\|\phi\| = \max_{-r\le s\le 0}|\phi(s)|,\ \phi\in C.$$

For $H > 0$ define $C_H \subset C$ by:

$$C_H = \{\phi\in C: \|\phi\| < H\}.$$

If $x:[-r, A) \to \Re^n$ is continuous, $0 < A \le \infty$, then, for each t in $[0, A)$, x_t in C is defined by:

$$x_t(s) = x(t + s),\ -r \le s \le 0,\ t \ge 0.$$

Let G be an open subset of C and consider the general autonomous delay differential system with finite delay:

$$\dot{x} = F(x_t),\ x_t = x(t + \theta),\ -r \le \theta \le 0,\ t \ge 0,$$

where $F:G \to \Re^n$ is a continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on F that each initial value problem:

$$\dot{x} = F(x_t),\ x_0 = \phi \in G,$$

has a unique solution defined on some interval $[0, A)$, $0 < A \le \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

Definition. The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \ge 0$. The zero solution is said to be unstable if it is not stable.

The main result of this paper is the following theorem.

Theorem 2. In addition to the basic assumptions imposed on A, H, G and F that appear in (3), we assume that there exist constants $a_1(< 0)$, $a_3(> 0)$ and $a_4(> 0)$ such that the following conditions hold:

$$\lambda_i(A) \le a_1,\ F(0) = 0,\ F(X) \ne 0,\ (X \ne 0),$$
$$0 < \lambda_i(JF(X)) \le a_4,$$

and $\lambda_i(G(X)) \ge a_3$ for all $X\in\Re^n$.

If $\tau < \dfrac{a_3}{\sqrt{n a_4}}$, then the solution $X = 0$ of (1) is unstable for arbitrary $H(Y)$.

Remark. It should be noted that there is no sign of restriction on eigenvalues of the matrix H in the system

(4) and it is clear that our assumptions have a very simple form and the applicability of them can be easily verified.

Proof. We define a Lyapunov-Krasovskii functional $V = V(X_t, Y_t, Z_t, W_t)$:

$$V(X_t,Y_t,Z_t,W_t) = -\langle AY,Z\rangle - \langle Y,W\rangle + \frac{1}{2}\langle Z,Z\rangle - \int_0^1 \sigma\langle H(\sigma Y)Y,Y\rangle d\sigma - \int_0^1 \langle F(\sigma X),X\rangle\, d\sigma - \lambda\int_{-\tau}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2\, d\theta\, ds.$$

It is clear that $V(0, 0, 0, 0) = 0$ and

$$V(0, 0, \varepsilon, 0) = \frac{1}{2}\langle\varepsilon,\varepsilon\rangle = \frac{1}{2}\|\varepsilon\|^2 > 0,$$

for all arbitrary $\varepsilon \neq 0$, $\varepsilon \in \Re^n$, which verifies the property (κ_1) of Krasovskii (1955).

Let $(X,Y, Z, W) = (X(t), Y(t), Z(t), W(t))$, be an arbitrary solution of (4). By the time derivative of V along system (4), we get:

$$\frac{d}{dt}V(X_t,Y_t,Z_t,W_t) = -\langle AZ,Z\rangle + \langle G(X)Y,Y\rangle + \langle F(X),Y\rangle + \langle H(Y)Z,Y\rangle + <\int_{t-\tau}^{t} J_F(X(s))Y(s)ds, Y> \frac{d}{dt}\int_0^1 \sigma\langle H(\sigma Y)Y,Y\rangle d\sigma - \frac{d}{dt}\int\langle F(\sigma X),X\rangle\, d\sigma - \frac{d}{dt}\lambda\int_{-\tau}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2 d\theta ds.$$

It can be easily seen that:

$$\frac{d}{dt}\int_0^1\langle F(\sigma X),X\rangle\, d\sigma = \langle F(X),Y\rangle,$$

$$\frac{d}{dt}\int_0^1 \sigma\langle H(\sigma Y),Y,Y\rangle\, d\sigma = \langle H(Y)Z,Y\rangle,$$

$$\frac{d}{dt}\int_{-\tau}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2\, d\theta\, ds = \|Y\|^2\tau - \int_{t-\tau}^{t}\|Y(\theta)\|^2 d\theta,$$

and

$$<\int_{t-\tau}^{t} J_F(X(s))Y(s)ds, Y> \geq -\|Y\|\left\|\int_{t-\tau}^{t} J_F(X(s))Y(s)ds\right\| \geq -\sqrt{n}a_4\|Y\|\left\|\int_{t-\tau}^{t} Y(s)\right\|ds$$

$$\geq -\sqrt{n}a_4\|Y\|\int_{t-\tau}^{t}\|Y(s)\|\, ds \geq -\frac{1}{2}\sqrt{n}a_4\tau\|Y\|^2 \quad -\frac{1}{2}\sqrt{n}a_4\int_{t-\tau}^{t}\|Y(s)\|^2\, ds,$$

so that

$$\frac{d}{dt}V(X_t,Y_t,Z_t,W_t) \geq -\langle AZ,Z\rangle + \langle G(X)Y,Y\rangle - \frac{1}{2}\sqrt{n}a_4\tau\langle Y,Y\rangle - \frac{1}{2}\sqrt{n}a_4\int_{t-\tau}^{t}\|Y(s)\|^2 ds - \lambda\tau\langle Y,Y\rangle + \int_{t-\tau}^{t}\|Y(\theta)\|^2\, d\theta$$

$$\geq -a_1\|Z\|^2 + \{a_3 - (\lambda + \frac{1}{2}\sqrt{n}a_4)\tau\}\|Y\|^2 + (\lambda - \frac{1}{2}\sqrt{n}a_4)\int_{t-\tau}^{t}\|Y(s)\|^2 ds.$$

Let

$$\lambda = \frac{1}{2}\sqrt{n}a_4.$$

Hence

$$\frac{d}{dt}V(X_t,Y_t,Z_t,W_t) \geq -a_1\|Z\|^2 + \left(a_3 - \sqrt{n}a_4\tau\right)\|Y\|^2.$$

If $\tau < \frac{a_3}{\sqrt{n}a_4}$, then we have for some positive constant k that:

$$\frac{d}{dt}V(X_t,Y_t,Z_t,W_t) \geq k\|Y\|^2 \geq 0,$$

which verifies the property (K_2) of Krasovskii (1955).

On the other hand, it follows that:

$$\frac{d}{dt}V(X_t,Y_t,Z_t,W_t) = 0 \Leftrightarrow Y = \dot{X},\ Z = \dot{Y} = 0,\ W = \dot{Z} = 0$$

for all $t \geq 0$.

Hence $X = \xi$, $Y = Z = W = 0$.

Substituting foregoing estimates in the system (4), we get that $F(\xi) = 0$, which necessarily implies that $\xi = 0$ since $F(0) = 0$. Thus, we have

$X = Y = Z = W = 0$ for all $t \geq 0$.

Hence, the property (K_3) of Krasovskii (1955) holds.

This completes proof of Theorem 2.

Example. In a special case of (3), for $n = 2$, we choose,

$$A = \begin{bmatrix} -6 & 2 \\ 2 & -6 \end{bmatrix},$$

$$G(X) = \begin{bmatrix} 4+\frac{1}{1+x_1^2} & 0 \\ 0 & 4+\frac{1}{1+x_2^2} \end{bmatrix},$$

and

$$F(X(t-\tau)) = \begin{bmatrix} 9x_1(t-\tau) \\ 9x_2(t-\tau) \end{bmatrix}.$$

Then, by an easy calculation, we obtain:

$$\lambda_1(A) = -4, \ \lambda_2(A) = -8,$$

$$\lambda_1(G) = 4 + \frac{1}{1+x_1^2},$$

and

$$\lambda_2(G) = 4 + \frac{1}{1+x_2^2}$$

$$J_F(X) = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix},$$

so that:

$$\lambda_i(A) \le -4 = a_1,$$

$$\lambda_i(G) \ge 4 = a_3 > 0,$$

and $0 < \lambda_i(J_H(X)) \le 5 = a_4, \quad (i = 1,2).$

Thus, all the conditions of Theorem 2 hold.

Conclusion

A nonlinear vector differential equation of the fourth order with constant deviating argument is considered. Based on the Krasovskii properties, the instability of the zero solution of the equation is discussed. In proving our result, we employ the Lyapunov-Krasovskii functional approach by defining a Lyapunov- Krasovskii functional. An example was given to illustrate the main result and concepts. The obtained result contributes and complements to previously known results on the qualitative behaviors of solutions in the literature.

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