

CERTAIN NEW CLASS OF CONCAVE FUNCTIONS DEFINED BY SĂLĂGEAN DIFFERENTIAL OPERATOR

(Kelas Baharu Tertentu bagi Fungsi Cekung yang Ditakrif oleh Pengoperasi Pembezaan Sălăgean)

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ABSTRACT

In this article, we introduce a class of concave analytic functions $S^n C_0(\alpha)$ defined by Sălăgean differential operator. We study some properties like coefficient inequalities, distortion theorems and extreme points for this class.

Keywords: analytic function; concave function; Sălăgean differential operator; coefficient inequalities; distortion theorem; extreme points

ABSTRAK

Dalam makalah ini, suatu kelas fungsi analisis cekung $S^n C_0(\alpha)$ yang ditakrif oleh pengoperasi pembezaan Sălăgean diperkenalkan. Beberapa sifat seperti ketaksamaan pekali, teorem erotan dan titik ekstrem untuk kelas ini dikaji.

Kata kunci: fungsi analisis; fungsi cekung; pengoperasi pembezaan Sălăgean; ketaksamaan pekali; teorem erotan; titik ekstrem

1. Introduction

The study of operators plays an important role in mathematics especially in geometric function theory. Recently, the interest in this area has been increasing because it permits detailed investigations of problems with physical applications. For example, the differential operator is linked between function theory and mathematical physics. Ruscheweyh (1975), Noor and Noor (1999, 2003), Noor (1999), Kim and Srivastava (1997); Darus and Ibrahim (2008) defined new operators and studied various classes of analytic and univalent functions, which generalise a number of previously well known subclasses and at that times they discovered new subclasses of analytic functions. Sălăgean (1983) defined the differential operator S^n of the class of analytic functions by using the technique of convolution. Many authors have used the Sălăgean operator to define and investigate the properties of certain known and new classes of analytic functions. We mentioned some of them in recent years.

Eker and Seker (2007) investigated new subclasses of multivalent functions involving Sălăgean operator and studied coefficient inequalities and other interesting properties of these classes.

Mahzoon and Latha (2009) studied new subclasses of analytic functions and derived some convolution properties of functions f in this class. Furthermore, Oros *et al.* (2009) introduced a class of holomorphic functions by using a certain operator S^n and obtained some subordination results.

In 2010, Li and Tang defined new classes of analytic functions by using the Sălăgean operator. They provided coefficient inequalities, distortion theorems, extreme points and radius of close-to-convexity, starlikeness and convexity of these classes. Moreover, Sun *et al.* (2010) derived some coefficient inequalities for certain classes of analytic functions, which were defined by means of the Sălăgean operator.

By using the Sălăgean differential operator and Alexander integral operator for analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ in the open unit disk D Kugita *et al.* (2010) defined the class of (α, δ) -neighbourhood for analytic functions with some interesting properties for (α, δ) -neighbourhood have been established.

Recently, Aouf *et al.* (2012) derived several subordination results for certain new classes of analytic functions defined by using S^n operator. El-Ashwah *et al.* (2013) gave some results for a new class of analytic functions defined by using Sălăgean operator and they studied some properties of functions in this class and obtained numerous sharp results including coefficient estimate, distortion theorem, radii of starlikeness, convexity, close-to-convexity, extreme points, integral means inequalities and partial sums of functions belonging to this class.

Also, Aouf *et al.* (2013) derived some results for certain subclasses of analytic functions defined by Sălăgean operator with varying arguments.

2. Preliminaries

Let H be the class of functions analytic in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and $H[\alpha, n]$ be the subclass of H consisting of all functions of the form

$f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \dots$. Let A be the subclass of H consisting of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in D. \tag{1}$$

For $f(z) \in A$, Sălăgean (1983) introduced the following operator, which is called the Sălăgean operator. This operator is defined as follows:

$$\begin{aligned} S^0 f(z) &= f(z), \\ S^1 f(z) &= z f'(z), \\ S^n f(z) &= S(S^{n-1} f(z)) \quad (n \in \mathbb{N} = 1, 2, \dots). \end{aligned}$$

We note that

$$S^n(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in N_0 = \mathbb{N} \cup \{0\}). \tag{2}$$

A function $f: D \rightarrow \mathbb{C}$ is said to be belonged to the family $C_0(\alpha)$, if f satisfies the following conditions:

- (1) f is analytic in D with the standard normalisation $f(0) = f'(0) - 1 = 0$. In addition, it satisfies $f(1) = \infty$.
- (2) f maps D conformally onto a set whose complement with respect to \mathbb{C} is convex.
- (3) The opening angle of $f(D)$ at ∞ is less than or equal to $\pi\alpha, \alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to concave univalent functions and for a detailed discussion about concave functions, we refer to Avkhadiev *et al.* (2006); Avkhadiev and Wirths (2005); Cruz and Pommerenke (2007) and the references therein.

Bhowmik *et al.* (2010) showed that an analytic function f maps D onto a concave domain of angle $\pi\alpha$ if and only if $\operatorname{Re} p_f(z) > 0, z \in D$, where

$$p_f(z) = \frac{2}{\alpha-1} \left[\frac{(\alpha+1)(1+z)}{2(1-z)} - 1 - z \frac{f''(z)}{f'(z)} \right].$$

Now in the following definition, we define new subclasses of concave analytic functions involving Sălăgean differential operator.

Definition 2.1. Let $f(z) \in A$. Then $f(z) \in S^n C_0(\alpha)$ if and only if

$$\operatorname{Re} \frac{2}{\alpha-1} \left[\frac{\alpha+1(1+z)}{2(1-z)} - 1 - z \frac{[S^n f(z)]''}{[S^n f(z)]'} \right] > 0, \quad (3)$$

$\alpha \in (1,2], n \geq 0$ and $z \in D$. Where $S^n f(z)$ is given by (2).

Remark 2.2. When $n=0$ in (3), we get the class of concave univalent functions.

This paper mainly studied the class $S^n C_0(\alpha)$. In particular, we provide coefficient inequalities, distortion inequalities and extreme points of the function in this class.

3. Coefficient inequalities for the class $S^n C_0(\alpha)$

In this section, we derive sufficient conditions for $f(z)$ belonging to the class $S^n C_0(\alpha)$, which are obtained by using coefficient inequalities.

Theorem 3.1. If $f(z) \in A$ satisfies the inequality

$$\sum_{k=2}^{\infty} ((\alpha-1)k^{n+1} + 2k^{n+2}) |a_k| < 3 - \alpha, \quad (4)$$

for some $\alpha \in (1,2], n \in \mathbb{N}$, then $f(z) \in S^n C_0(\alpha)$.

Proof. We want to prove that

$$\operatorname{Re} \frac{2}{\alpha-1} \left[\frac{\alpha+1(1+z)}{2(1-z)} - \frac{zg'(z)}{g(z)} \right] > 0.$$

By using the fact that $\operatorname{Re} \frac{1}{w} > \frac{1}{2} \Leftrightarrow |w-1| < 1$, it is enough to show that $|w| < 1$.

$$\frac{1}{w} = \frac{2}{\alpha-1} \left[\frac{\alpha+1}{2} \frac{1+z}{1-z} - \frac{zg'(z)}{g(z)} \right], \quad (5)$$

where $g(z) = z(S^n f(z))'$ and $S^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$.

So we have

$$\begin{aligned} g(z) &= z \left(1 + \sum_{k=2}^{\infty} k^{n+1} a_k z^{k-1} \right), \\ g'(z) &= 1 + \sum_{k=2}^{\infty} k^{n+2} a_k z^{k-1}. \end{aligned}$$

Substituting for $g(z)$ and $g'(z)$ in (5) yields

$$|w| = \frac{\alpha-1}{2} \frac{\left| 2(1-z)(z) \left(1 + \sum_{k=2}^{\infty} k^{n+1} a_k z^{k-1} \right) \right|}{\left| (\alpha+1)(1+z)(z) \left(1 + \sum_{k=2}^{\infty} k^{n+1} a_k z^{k-1} \right) - 2(z)(1-z) \left(1 + \sum_{k=2}^{\infty} k^{n+2} a_k z^{k-1} \right) \right|}.$$

Let $z \rightarrow -1$, we observe that

$$\begin{aligned} |w| &\leq \frac{\alpha-1}{2} \left(\frac{1 + \sum_{k=2}^{\infty} k^{n+1} |a_k|}{1 + \sum_{k=2}^{\infty} k^{n+2} |a_k|} \right) \\ &< \frac{\alpha-1}{2} \left(\frac{1 + \sum_{k=2}^{\infty} k^{n+1} |a_k|}{1 - \sum_{k=2}^{\infty} k^{n+2} |a_k|} \right). \end{aligned}$$

The last expression is bounded by 1, if

$$\frac{1 + \sum_{k=2}^{\infty} k^{n+1} |a_k|}{1 - \sum_{k=2}^{\infty} k^{n+2} |a_k|} \leq \frac{2}{\alpha-1},$$

which is equivalent to the condition (4). This completes the proof of Theorem 3.1. \square

4. Distortion Inequalities

Using the inequality (4), the following result gives the distortion bounds for functions in the class $S^n C_0(\alpha)$.

Lemma 4.1. *If $f(z) \in S^n C_0(\alpha)$, then we have*

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{(3-\alpha) - \sum_{k=2}^p ((\alpha-1)k^{n+1} + 2k^{n+2}) a_k}{(\alpha-1)(p+1)^{n+1} + 2(p+1)^{n+2}}, \quad (6)$$

where $n \in \mathbb{N}$, $\alpha \in (1, 2]$.

Proof. In view of Theorem 3.1, we can write

$$\sum_{k=p+1}^{\infty} ((\alpha-1)k^{n+1} + 2k^{n+2}) a_k \leq (3-\alpha) - \sum_{k=2}^p ((\alpha-1)k^{n+1} + 2k^{n+2}) a_k.$$

Thus, we obtain

$$(\alpha-1)(p+1)^{n+1} + 2(p+1)^{n+2} \sum_{k=p+1}^{\infty} a_k \leq (3-\alpha) - \sum_{k=2}^p ((\alpha-1)k^{n+1} + 2k^{n+2}) a_k.$$

Hence this implies

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{(3-\alpha) - \sum_{k=2}^p ((\alpha-1)k^{n+1} + 2k^{n+2}) a_k}{(\alpha-1)(p+1)^{n+1} + 2(p+1)^{n+2}} = \Upsilon_k. \square$$

Lemma 4.2. If $f(z) \in S^n C_0(\alpha)$, then we have

$$\sum_{k=p+1}^{\infty} k a_k \leq \frac{(3-\alpha) - \sum_{k=2}^p ((\alpha-1)k^{n+1} + 2k^{n+2}) a_k}{(\alpha-1)(p+1)^n + 2(p+1)^{n+1}} = \Lambda_k,$$

where $n \in \mathbb{N}$, $\alpha \in (1, 2]$.

Proof. In view of Theorem 3.1, we can write

$$\sum_{k=p+1}^{\infty} ((\alpha-1)k^n + 2k^{n+1}) k a_k \leq (3-\alpha) - \sum_{k=2}^p ((\alpha-1)k^{n+1} + 2k^{n+2}) a_k.$$

Thus, we obtain

$$(\alpha-1)(p+1)^n + 2(p+1)^{n+1} \sum_{k=p+1}^{\infty} k a_k \leq (3-\alpha) - \sum_{k=2}^p ((\alpha-1)k^{n+1} + 2k^{n+2}) a_k.$$

Hence this implies

$$\sum_{k=p+1}^{\infty} k a_k \leq \frac{(3-\alpha) - \sum_{k=2}^p ((\alpha-1)k^{n+1} + 2k^{n+2}) a_k}{(\alpha-1)(p+1)^n + 2(p+1)^{n+1}} = \Lambda_k. \square$$

Theorem 4.3. Let $f(z) \in S^n C_0(\alpha)$. Then, for $|z| \rightarrow 1$ we have

$$1 - \sum_{k=2}^p a_k |z|^k - \Upsilon_k \leq |f(z)| \leq 1 + \sum_{k=2}^p a_k |z|^k + \Upsilon_k,$$

and

$$1 - \sum_{k=2}^p k a_k |z|^{k-1} - \Lambda_k \leq |f'(z)| \leq 1 + \sum_{k=2}^p k a_k |z|^{k-1} + \Lambda_k.$$

Proof. Let $f(z)$ given by (1). For $|z| \rightarrow 1$ and using Lemma 4.1, we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{k=2}^p a_k |z|^k + \sum_{k=p+1}^{\infty} a_k |z|^k \leq |z| + \sum_{k=2}^p a_k |z|^k + |z|^{p+1} \sum_{k=p+1}^{\infty} a_k \\ &\leq 1 + \sum_{k=2}^p a_k |z|^k + \Upsilon_k. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{k=2}^p a_k |z|^k - \sum_{k=p+1}^{\infty} a_k |z|^k \geq |z| - \sum_{k=2}^p a_k |z|^k - |z|^{p+1} \sum_{k=p+1}^{\infty} a_k \\ &\geq 1 - \sum_{k=2}^p a_k |z|^k - \Upsilon_k. \end{aligned}$$

Also, for $|z| \rightarrow 1$ and using Lemma 4.2, we obtain

$$\begin{aligned} |f'(z)| &\leq 1 + \sum_{k=2}^p k a_k |z|^{k-1} + \sum_{k=p+1}^{\infty} k a_k |z|^{k-1} \leq 1 + \sum_{k=2}^p k a_k |z|^{k-1} + |z|^p \sum_{k=p+1}^{\infty} k a_k \\ &\leq 1 + \sum_{k=2}^p k a_k |z|^{k-1} + \Lambda_k \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq 1 - \sum_{k=2}^p k a_k |z|^{k-1} - \sum_{k=p+1}^{\infty} k a_k |z|^{k-1} \geq 1 - \sum_{k=2}^p k a_k |z|^{k-1} - |z|^p \sum_{k=p+1}^{\infty} k a_k \\ &\geq 1 - \sum_{k=2}^p k a_k |z|^{k-1} - \Lambda_k. \end{aligned}$$

This completes the assertion of Theorem 4.3. \square

Taking $p=1$ in Theorem 4.3, we have the following result.

Corollary 4.4. *Let $f(z) \in S^n C_0(\alpha)$. Then for $|z| \rightarrow 1$*

$$1 - \frac{3-\alpha}{2^{n+1}(\alpha+3)} \leq |f(z)| \leq 1 + \frac{3-\alpha}{2^{n+1}(\alpha+3)}$$

and

$$1 - \frac{3-\alpha}{2^n(\alpha+3)} \leq |f'(z)| \leq 1 + \frac{3-\alpha}{2^n(\alpha+3)}.$$

5. Extreme points

The determination of the extreme points of a family $f(z)$ of concave functions enables to solve many extreme problems of $f(z)$. Now, let us determine extreme points of the class $S^n C_0(\alpha)$.

Theorem 5.1. Let $f_1(z) = z$ and

$$f_k(z) = z + \frac{3-\alpha}{(\alpha-1)k^{n+1} + 2k^{n+2}} z^k \quad (k=2,3,\dots),$$

where $\alpha \in (1,2]$, and $n \in \mathbb{N}$.

Then, $f(z) \in S^n C_0(\alpha)$ if and only if it can be expressed in the form

$$f(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z),$$

where $\mu_k > 0$ and $\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k$.

Proof. Suppose that

$$f(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z) = z + \sum_{k=2}^{\infty} \mu_k \frac{3-\alpha}{(\alpha-1)k^{n+1} + 2k^{n+2}} z^k.$$

Then

$$\sum_{k=2}^{\infty} \left((\alpha-1)k^{n+1} + 2k^{n+2} \right) \frac{3-\alpha}{((\alpha-1)k^{n+1} + 2k^{n+2})} \mu_k = \sum_{k=2}^{\infty} (3-\alpha) \mu_k = (3-\alpha)(1-\mu_1) < 3-\alpha.$$

Thus, $f(z) \in S^n C_0(\alpha)$ from the definition of the class of $f(z) \in S^n C_0(\alpha)$.

Conversely, suppose that $f(z) \in S^n C_0(\alpha)$. Since

$$a_k \leq \frac{3-\alpha}{(\alpha-1)k^{n+1} + 2k^{n+2}} \quad (k=2,3,\dots),$$

we may set that

$$\mu_k = \frac{(\alpha-1)k^{n+1} + 2k^{n+2}}{3-\alpha} a_k \quad (k=2,3,\dots)$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k.$$

So that

$$f(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z) = z + \sum_{k=2}^{\infty} \mu_k \frac{3-\alpha}{(\alpha-1)k^{n+1} + 2k^{n+2}} z^k.$$

This completes the proof of Theorem 5.1. \square

Corollary 5.2. *The extreme points of $S^n C_0(\alpha)$ are the functions $f_1(z) = z$ and*

$$f_k(z) = z + \frac{3-\alpha}{(\alpha-1)k^{n+1} + 2k^{n+2}} z^k \quad (k = 2, 3, \dots).$$

6. Conclusion

The class of concave analytic functions $S^n C_0(\alpha)$ defined by Sălăgean differential operator is introduced. Some characterisations and properties of this class are investigated.

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