A Third Order Nakashima Type Implicit Pseudo Runge-Kutta Method for Delay Differential Equations
(Kaedah Runge-Kutta Nakashima Jenis ‘Pseudo’ Bertahap Tiga untuk Persamaan Perbezaan Lengah)

LIM TIAN HWE* 

ABSTRACT
A third order Nakashima type implicit Pseudo Runge-Kutta method is presented. The free parameter was determined by minimizing the error bound. The stability region of the method was presented. Some problems on delay differential equations are tested to compare the accuracy of the proposed method with third order RADAU I.

Keywords: Delay differential equations; implicit pseudo Runge-Kutta method; third order

ABSTRAK

Kata kunci: Bertahap tiga; kaedah Runge-Kutta jenis ‘pseudo’ tersirat; persamaan perbezaan lengah

INTRODUCTION
The first order delay differential equation (DDE) with one delay term can be written as:

\[
\begin{align*}
    y'(t) &= f(t, y(t), y(t - \tau)) \\
    y(t) &= \varphi(t) \quad t \neq t_n
\end{align*}
\]

where \(\varphi(t)\) is the initial function. \(\tau\) is, in term of \(t\) and \(y(t)\), called the delay argument and \(y(t - \tau)\) is the solution of the delay term.

In recent years, research has been carried out to solve DDEs using explicit or implicit Runge-Kutta (RK) method with Hermite interpolation. Such work can be found in Bellen and Zennaro (2003), Ismail et al (2002), Karoui (1992) and Orbele and Pesch (1981). However, Yaacob et al. (2011) studied the numerical treatment of DDEs using Pseudo Runge-Kutta method (PRK).

In this paper, we discussed the derivation of a third order implicit PRK method in the next section. The proposed method is compared with other third order implicit method.

IMPLICIT PSEUDO RUNGE-KUTTA METHOD
Nakashima (1982) introduce a new type of PRK method. The PRK can be written as:

\[
\begin{align*}
    y_{n+1} &= y_n + h \sum_{i=0}^{s} b_i K_i \\
    K_i &= f(t_n + c_i h, y_n + \lambda_i (y_n - y_{n-i}) + h \sum_{j=0}^{s} a_{ij} K_j) \\
    c_i &= \lambda_i + \sum_{j=0}^{s} a_{ij} \quad i = 2, \ldots, s, \\
    \lambda_i &= -1, \quad \lambda_s = 0
\end{align*}
\]

for \(c_0 = \lambda_0 = -1, \; c_s = \lambda_s = 0\) and \(0 \leq c_i \leq 1\).

From Nakashima (1982) and Shintani (1981), the following eight order conditions, which are required to construct a third order PRK method.

\[
\begin{align*}
    \tau^{(0)}_1 &\;: \; \sum b_i = 1 \\
    \tau^{(1)}_2 &\;: \; \sum b_i c_i = \frac{1}{2} \\
    \tau^{(2)}_3 &\;: \; \sum b_i c_i^2 = \frac{1}{4} \\
    \tau^{(3)}_4 &\;: \; \sum b_i a_i = \frac{1}{4}
\end{align*}
\]

From Table 1, there are only four equations to be satisfied and we have 6 unknowns, thus we have two arbitrary parameters to determine. After solving all the related equations, we have:

\[
\begin{align*}
    b_0 &= \frac{1}{6} \frac{3c_2 - 2}{c_2 + 1}, \quad b_1 = \frac{1}{6} \frac{9c_2 - 5}{c_2} \\
    b_2 &= \frac{5}{6} \frac{1}{c_2 (c_2 + 1)}, \quad \lambda_2 = -c_2^2 - 2a_{20} + 2a_{22} c_2.
\end{align*}
\]

where \(c_2, a_{20}\) and \(a_{22}\) are free parameters which we want to determine.
All four error factors will become zero for \( c_2 = 0.7 \), \( a_{20} = 0.833 \) and \( a_{22} = 0 \). However, this also means for this choice of \( c_2 \), \( a_{20} \) and \( a_{22} \), our third order method would actually become a fourth order method. However, by choosing \( c_2 \) for a value close to 0.7, such as \( c_2 = \frac{4}{5} \), we expect that the error factor will become small.

We use the notation from Lotkin (1951) and Ralston (1962) to determine the error bounds \( E \) for our third order PRK method.

\[
|E| \leq C M L^2 h^2, \tag{4}
\]

where \( C \) is the error constant in a region \( \mathbb{R} \) about \((t, y)\)

\[
\left| f(x, y) \right| < M \text{ and } \left| \frac{\partial f(x, y)}{\partial y'} \right| < \frac{M}{M''}, \tag{5}
\]

where \( L \) and \( M \) are positive constants independent of \( t, y \).

With \( c_2 = \frac{4}{5} \), the constant \( C \) is estimated by

\[
C = \frac{5}{846} \left( \frac{7}{18} + \frac{2209}{4752} a_{20} + \frac{193}{144} a_{22} \right)
+ \left( \frac{7}{36} + \frac{9504}{9504} a_{20} + \frac{93}{288} a_{22} \right) \tag{6}
\]

Our objective was to minimize the right hand side of (6). We found that the bound of \( C \) is minimized when \( a_{20} = \frac{3}{118} \) and \( a_{22} = \frac{3}{118} \). Substituting \( c_2, a_{20} \) and \( a_{22} \) into (3), we obtain a two-stages third order pseudo Runge-Kutta method:

\[
y(t_i + h) = y_i + h \left( -\frac{4}{5} k_0 + \frac{3}{118} k_2 + \frac{2209}{426337} k_1 \right),
\]

where

\[
k_0 = f(t_i, y_i),
\]

\[
k_1 = f(t_i + \frac{3}{47} h, y_i - \frac{36465}{426337} (y_i - y_i)),
\]

\[
k_2 = f\left(t_i + \frac{212104}{426337} h, y_i - \frac{56}{193} (y_i - y_i)\right) \tag{7}
\]

We present our new implicit PRK method in the tableau.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( a_{20} )</th>
<th>( a_{22} )</th>
<th>( a_{20} )</th>
<th>( a_{22} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>(-\frac{1}{4})</td>
<td>(\frac{1}{5})</td>
<td>(\frac{3}{118})</td>
<td>(\frac{3}{118})</td>
</tr>
</tbody>
</table>

The local truncation error for formula (7) satisfies:

\[
|E| \leq \frac{1}{20} M L^2 h^2. \tag{8}
\]

**STABILITY ANALYSIS**

To determine the stability function of the proposed method, we applied the famous Dahlquist’s test equation:

\[
y' = f(x, y) = \lambda y, \tag{9}
\]

to formula (7). The stability polynomial for the proposed pseudo-Runge-Kutta method is:

\[
y_{n+1} = -\frac{\lambda h}{193 + 56h\lambda} \begin{pmatrix}
2316 y_n + 1530 h\lambda y_n + 114 h\lambda^2 y_n + \\
47 h^3 y_n + 5936 h^2 y_n + 193 + 56 h\lambda
\end{pmatrix}. \tag{10}
\]

On assuming \( h\lambda = z, y_{n+1} = \zeta, y_n = \xi^0 \) and \( y_{n+1} = \xi^{-1} \), (10) becomes:

\[
\zeta + \frac{\lambda h + 114z + 7z^2 + 593z^3}{12(-193 + 56h\lambda)} = 0. \tag{11}
\]

Solving (11) we have two roots

\[
\zeta_1 = \frac{-2316 - 1530z - 593z^2 + \sqrt{\sigma}}{2(-2316 + 672z) \tag{12}
\]

\[
\zeta_2 = \frac{-2316 - 1530z - 593z^2 - \sqrt{\sigma}}{2(-2316 + 672z) \tag{12}
\]

where

\[
\sigma = 5363856 + 8143056 \xi + 4846092 \xi^2 + 1795764 \xi^3 + 351649 \xi^4.
\]

Since \( |\zeta_1| \leq 1 \) and \( |\zeta_2| \leq 1 \) are two stability functions for the new PRK method. By taking \( z = x + yi \), we plot the stability region using MAPLE package. The shaded region is the region which satisfies the condition \( |\zeta_1| \leq 1 \) and \( |\zeta_2| \leq 1 \). The stability region for the new method is given in Figure 1.
NUMERICAL METHOD FOR SOLVING DDES

Since the method we proposed in earlier section with a third order two stages method, we compare our methods with a two stages implicit RK method show in Table 3.

Problem 2: (Ismail et al. 2002)
\[ y'(t) = \cos(t) \, y(t-\tau), \quad t > 0 \]
\[ y(t) = 1, \quad t \leq 0 \]
\[ \tau(t,y) = t - y(t) + 2 \]
Exact solution: \( y(t) = \sin(t) + 1 \)
Results are given for \( t \in [0,10] \)

Problem 3: (Bellen & Zennaro 2003)
\[ y'(t) = \frac{1}{2\sqrt{t}} y(t-\tau), \quad t > 1 \]
\[ y(t) = 1, \quad t \leq 1 \]
\[ \tau(t,y) = y(t) - \sqrt{2} + 1 \]
Exact solution: \( y(t) = \sqrt{t}, \quad 1 \leq t \leq 2 \)
Results are given for \( t \in [1,2] \)

We solved the above delay differential equations using PRK proposed in the previous section and an implicit Runge-Kutta method describe in Table 4 with tolerance, \( TOL=0.01 \) and 5 Newton iterations to get the stages values. The delay term is evaluated using 3 point Hermite interpolation. The notations used are as follows:

- \( H \) : Stepsize
- \( M1 \) : Using Implicit PRK in Table 3
- \( M2 \) : Using RADAU I in Table 4
- \( \text{MaxErr} \) : Maximum error \( |y(x) - y_i| \)

The notation 6.2823577 (-7) means 6.2823577 \( \times 10^{-7} \).

<table>
<thead>
<tr>
<th>H</th>
<th>Method</th>
<th>MaxErr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>M1</td>
<td>4.5696171 (-7)</td>
</tr>
<tr>
<td></td>
<td>M2</td>
<td>7.7241505(-6)</td>
</tr>
<tr>
<td>0.01</td>
<td>M1</td>
<td>4.9170357(-10)</td>
</tr>
<tr>
<td></td>
<td>M2</td>
<td>7.7795651(-9)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>H</th>
<th>Method</th>
<th>MaxErr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>M1</td>
<td>6.2935352 (-6)</td>
</tr>
<tr>
<td></td>
<td>M2</td>
<td>9.2600280 (-6)</td>
</tr>
<tr>
<td>0.01</td>
<td>M1</td>
<td>5.9143795 (-10)</td>
</tr>
<tr>
<td></td>
<td>M2</td>
<td>9.2592766 (-9)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>H</th>
<th>Method</th>
<th>MaxErr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>M1</td>
<td>1.4383307 (-6)</td>
</tr>
<tr>
<td></td>
<td>M2</td>
<td>3.1670842 (-7)</td>
</tr>
<tr>
<td>0.01</td>
<td>M1</td>
<td>6.1223481 (-11)</td>
</tr>
<tr>
<td></td>
<td>M2</td>
<td>1.4304856 (-9)</td>
</tr>
</tbody>
</table>
The notation 6e-07 means 6 × 10^{-6}.

**CONCLUSION**

In this paper, we derived an implicit third order PRK to solve DDEs. The proposed method is then compared with RADAU I. The delay term is approximated using 3-point Hermite interpolation. From all of the problems we test, the proposed method give better results than RADAU I.

**REFERENCES**


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