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A COMPARISON BETWEEN THE PERFORMANCES OF DOUBLE SAMPLING $\bar{X}$ AND VARIABLE SAMPLE SIZE $\bar{X}$ CHARTS

(Suatu Perbandingan antara Prestasi Carta-carta $\bar{X}$ Pensampelan Berganda dan $\bar{X}$ dengan Saiz Sampel yang Berubah-ubah)

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ABSTRACT

The double sampling (DS) $\bar{X}$ and variable sample size (VSS) $\bar{X}$ charts are very effective to detect small and moderate shifts in the process mean. Both charts are usually investigated under the assumption of known process parameters. However, the process parameters are commonly estimated from an in-control Phase-I dataset because they are usually unknown in practice. Therefore, both cases of known and estimated process parameters for the DS $\bar{X}$ and VSS $\bar{X}$ charts are considered in this paper. It is well known that the run length distribution of a control chart is highly skewed, especially when the process parameters are estimated and the process is in-control or slightly out-of-control. Interpretation based solely on a specific performance measure could be misleading. Thus, various performance measures need to be used to evaluate the properties of the control charts. Generally, the design of a control chart with estimated process parameters is proposed without comparing with other control charts. Accordingly, this paper focuses mainly on the comparison of the average run length (ARL), standard deviation of the run length (SDRL) and average sample size (ASS) between the DS $\bar{X}$ and VSS $\bar{X}$ charts with known and estimated process parameters. The ARL and SDRL results indicate that the DS $\bar{X}$ chart outperforms the VSS $\bar{X}$ chart for all ranges of shifts. However, the converse is true in terms of the ASS.

Keywords: double sampling (DS) $\bar{X}$ chart; variable sample size (VSS) $\bar{X}$ chart; average run length (ARL); standard deviation of the run length (SDRL); average sample size (ASS)

ABSTRAK

Carta $\bar{X}$ pensampelan berganda (DS) dan carta $\bar{X}$ dengan saiz sampel yang berubah-ubah (VSS) adalah sangat berkesan untuk mengesan anjakan min proses yang kecil dan sederhana. Kedua-dua carta ini biasanya disiasat dengan andaian bahawa parameter-parameter proses adalah diketahui. Walau bagaimanapun, parameter-parameter proses biasanya dianggarkan daripada set data Fasa-I yang berada dalam kawalan kerana parameter-parameter proses biasanya tidak diketahui dalam amalan. Oleh hal yang demikian, kedua-dua kes dengan parameter-parameter proses yang diketahui dan dianggarkan bagi carta-carta $\bar{X}$ DS dan $\bar{X}$ VSS dipertimbangkan dalam makalah ini. Adalah diketahui bahawa taburan panjang larian bagi suatu carta kawalan adalah sangat terpencong, terutamanya apabila parameter-parameter proses dianggarkan dan proses berada dalam kawalan atau hanya sedikit yang berada di luar kawalan. Tafsiran yang semata-mata berdasarkan satu ukuran prestasi yang spesifik adalah mengelirukan. Justeru, pelbagai ukuran prestasi perlu digunakan untuk menilai sifat-sifat carta kawalan. Secara umumnya, reka bentuk carta kawalan berdasarkan penganggaran parameter proses dicadangkan tanpa perbandingan dengan carta-carta kawalan yang lain. Makalah ini bertujuan untuk membandingkan panjang larian purata (ARL), sisihan piawai panjang larian (SDRL) dan saiz sampel purata (ASS) antara carta-carta $\bar{X}$ DS dan $\bar{X}$ VSS berdasarkan parameter-parameter proses yang diketahui dan dianggarkan. Keputusan ARL dan SDRL menunjukkan bahawa carta $\bar{X}$ DS adalah lebih baik daripada carta $\bar{X}$ VSS bagi semua julat anjakan. Namun demikian, hal yang sebaliknya adalah benar jika dikaji dari segi ASS.

Kata kunci: carta $\bar{X}$ pensampelan berganda (DS); carta $\bar{X}$ dengan saiz sampel yang berubah-ubah (VSS); panjang larian purata (ARL); sisihan piawai panjang larian (SDRL); saiz sampel purata (ASS)

1. Introduction

Statistical Process Control (SPC) is an effective problem-solving technique to ameliorate process capability and attain process stability via the reduction of variability. Control chart is a very useful technique in many industries. In recent years, studies of adaptive control charts become more popular among researchers than that of the static control charts because the static control charts are less sensitive in responding to process changes. Adaptive control charts allow the charts' parameters, which include the sample size, sampling interval and control limits, to vary at different states (Castagliola *et al.* 2013). Recent works that deal with adaptive charts, such as double sampling (DS), variable sample size (VSS), variable sampling interval (VSI) and variable sample size and sampling interval (VSSI) charts, can be found in Amiri *et al.* (2014), Costa and De Magalhães (2007), De Magalhães *et al.* (2009), Mahadik (2013) and Teoh *et al.* (2014). The DS $\bar{X}$ and VSS $\bar{X}$ charts are adaptive control charts that are sensitive for the detection of small to moderate mean shifts in the process. Since only one chart's parameter, i.e the sample size, varies for these two charts, both the DS $\bar{X}$ and VSS $\bar{X}$ charts are studied in this paper in order to make fair comparison.

In real-life applications, the process parameters are estimated from an in-control Phase-I dataset because they are normally unknown. The performance of the control chart with estimated process parameters is significantly different from that of the known-process-parameter case. Therefore, numerous researchers (Capizzi & Masarotto 2010; Khoo *et al.* 2013a; Mahmoud & Maravelakis 2010; Testik 2007) studied the impact of estimations of process parameters on a variety of control charts' performances. Jensen *et al.* (2006) and Psarakis *et al.* (2014) provided thorough reviews on the recent developments of process parameters estimation on various types of control charts. The accuracy of the estimated process parameters determined from the Phase-I dataset is critical to ensure a favourable performance in the Phase-II process. Thus, some researchers (Castagliola *et al.* 2012; Maravelakis & Castagliola 2009; Teoh *et al.* 2014; Zhang *et al.* 2011) recently implemented new and optimal charting parameters, specially designed for the control charts with estimated process parameters. Moreover, Dasgupta and Mandal (2008) applied the Bayesian approach to process parameter estimation and used it to obtain the optimal diagnosis interval for detecting the occurrence of assignable cause in the process.

The DS $\bar{X}$ chart, which follows the idea of the double sampling plan, was presented by Daudin (1992) to overcome the setback of the Shewhart $\bar{X}$ chart towards small process shifts. There are many literature focusing on the DS $\bar{X}$ type charts for monitoring the process mean, such as those by Carot *et al.* (2002), Claro, *et al.* (2008) and Khoo *et al.* (2011). Torng *et al.* (2009) formulated an economic-statistical-design model to reduce the total cost of the DS $\bar{X}$ chart. They also applied the genetic algorithm to determine the chart's optimal parameters. They claimed that the DS $\bar{X}$ chart is favoured for enhancing the effectiveness of process monitoring without increasing the number of samples. Also, it maintains the simplicity of obtaining the $\bar{X}$ chart's statistic. The performance of the DS $\bar{X}$ control chart under non-normality was studied by Torng and Lee (2009). They showed that the DS $\bar{X}$ chart is equally competitive as the variable parameter (VP) $\bar{X}$ chart and surpasses the Shewhart $\bar{X}$ chart in terms of the efficiency in detecting small mean shifts. Costa and Machado (2011) used the Markov chain approach to analyse the performance of the VP $\bar{X}$ and DS $\bar{X}$ charts in the existence of correlation. While so much work focused on the DS type control chart with known process parameters, Khoo *et al.* (2013b) and Teoh *et al.* (2014) recently proposed the DS $\bar{X}$ chart with estimated process parameters. Khoo *et al.* (2013b) introduced three optimal design procedures of the ARL-based DS $\bar{X}$ chart with estimated process parameters. Teoh *et al.* (2014) on the other hand, proposed a new optimal design procedure for minimising the out-of-control median run length.

Prabhu *et al.* (1993) and Costa (1994) used the Markov chain approach to evaluate the VSS $\bar{X}$ chart. The VSS $\bar{X}$ chart has a significant improvement for detecting small process shifts compared to the Shewhart $\bar{X}$ chart (Prabhu *et al.* 1993). Costa (1994) claimed that the VSS $\bar{X}$ chart has some advantages over the VSI $\bar{X}$ chart, EWMA chart, CUSUM chart and the $\bar{X}$ chart with supplementary runs rules for some ranges of shifts. Park and Reynolds (1994) and Kooli and Limam (2011) formulated an economic design for minimising the expected cost per hour for the VSS $\bar{X}$ and VSS *np* charts, respectively. They found that the VSS type control charts provide more cost savings compared to the static control charts. Because of the merits of the VSS properties, Wu (2011) examined the expected long-run cost per unit time for a three-state monitoring system by applying the VSS control chart. Castagliola *et al.* (2013) discussed the VSS *t* control chart for observing the short runs process. For attribute control charts, Luo and Wu (2002) developed the optimal VSS *np* and VSI *np* charts for fraction nonconforming. For adaptive EWMA and CUSUM type charts, Zhang and Wu (2007) introduced the VSS weighted loss function CUSUM scheme to improve the detection of a broad domain of mean shifts and increasing variance shifts. To improve the efficiency of the EWMA control chart, Amiri *et al.* (2014) and Zhang and Song (2014) proposed a new VSS EWMA chart with the application of integer linear function and the VSS EWMA median chart, respectively. Note that all the aforementioned literature only considers the VSS type charts with known process parameters. Recently, Castagliola *et al.* (2012) extended Costa's (1994) work by developing an optimal design of the VSS $\bar{X}$ chart with estimated process parameter.

To date, none of the existing literature compares the performances of different control charts with estimated process parameters. It is well known that the run length distribution of a control chart is highly skewed, especially when the process parameters are estimated (Jensen *et al.* 2006; Jones *et al.* 2004; Teoh *et al.* 2014). Therefore, various performance measures should be used to evaluate a control chart. Thus, this paper aims at providing comprehensive comparative studies based on various performance criteria, i.e. the average run length (ARL), standard deviation of the run length (SDRL) and average sample size (ASS) of the DS $\bar{X}$ and VSS $\bar{X}$ charts with estimated process parameters.

The structure for the remainder of the paper is as follows: Sections 2 and 3 deal with the DS $\bar{X}$ and VSS $\bar{X}$ charts, respectively, with their run length properties for both cases of known and estimated process parameters. Section 4 compares the DS $\bar{X}$ and VSS $\bar{X}$ charts based on the ARL, SDRL and ASS, for the known- and estimated-process-parameter cases. A conclusion is presented in Section 5.

2. The DS $\bar{X}$ Chart

Assume that the Phase-II observations, Y, of a quality characteristic are independent and identically distributed (iid) normal $N\left(\mu_0, \sigma_0^2\right)$ random variables, where μ_0 and σ_0^2 are the in-control mean and variance, respectively. By referring to Figure 1(a), the DS $\bar{X}$ chart is divided into distinct portions denoted by $I_1 = \left[-L_1, L_1\right]$, $I_2 = \left[-L, -L_1\right) \cup \left(L_1, L\right]$, $I_3 = \left(-\infty, -L\right) \cup \left(L, +\infty\right)$ and $I_4 = \left[-L_2, L_2\right]$. Note that $L_1 > 0$ is the warning limit in the first-sample stage; while $L \geq L_1$ and $L_2 > 0$ are the control limits in the first-sample and combined-sample stages, respectively.

The DS $\bar{X}$ chart is implemented by determining the limits; L, L_1 and L_2. The construction of the control chart is then followed by taking a first sample of size n_1 and then compute out the sample mean $\bar{Y}_{1k} = \sum_{j=1}^{n_1} Y_{1k,j} / n_1$. Here, $Y_{1k,j}$, for $j = 1, 2, \ldots, n_1$ represents the Phase-II observations of the first sample. Then calculate $Z_{1k} = \left[\left(\bar{Y}_{1k} - \mu_0\right)\sqrt{n_1}\right] / \sigma_0$. The process is considered as in-control when $Z_{1k} \in I_1$; while the process is considered as out-of-control when $Z_{1k} \in I_3$. Besides, the second sample of size n_2 needs to be taken from the same population as that of the first sample when $Z_{1k} \in I_2$. This is followed by computing the second sample mean $\bar{Y}_{2k} = \sum_{j=1}^{n_2} Y_{2k,j} / n_2$, where $Y_{2k,j}$, for $j = 1, 2, \ldots, n_2$, are the Phase-II observations of the second sample. Next, obtain the combined-sample mean $\bar{Y}_k = \left(n_1 \bar{Y}_{1k} + n_2 \bar{Y}_{2k}\right) / \left(n_1 + n_2\right)$. If $Z_k = \left[\left(\bar{Y}_k - \mu_0\right)\sqrt{n_1 + n_2}\right] / \sigma_0 \in I_4$, the process is proclaimed as in-control; otherwise, the process is declared as out-of-control.

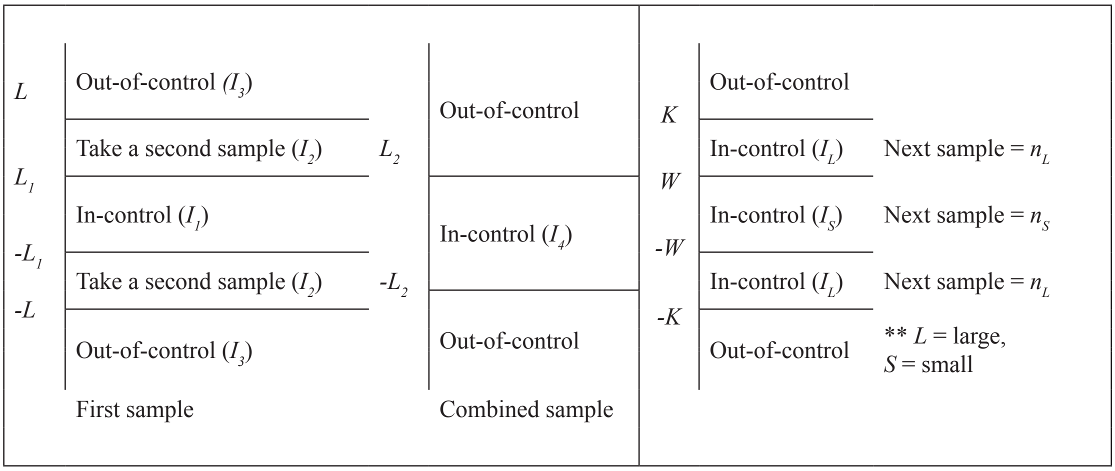
Let P_{a1} and P_{a2} be the probabilities of declaring an in-control process for the first sample and after taking the second sample, respectively. According to Daudin (1992), the probability that the process is regarded as in-control can be expressed as $P_a = P_{a1} + P_{a2}$, where

$$P_{a1} = \Phi\left(L_1 + \delta\sqrt{n_1}\right) - \Phi\left(-L_1 + \delta\sqrt{n_1}\right) \tag{1}$$

and

$$P_{a2} = \int_{Z \in I_2 *} \left[\Phi\left(cL_2 + rc\delta - \sqrt{\frac{n_1}{n_2}} z\right) - \Phi\left(-cL_2 + rc\delta - \sqrt{\frac{n_1}{n_2}} z\right)\right] \phi(z) dz. \tag{2}$$

The symbols $\Phi(\cdot)$ and $\phi(\cdot)$ shown in Equations (1) and / or (2) represent the standard normal



(a) (b)

Figure 1: Graphical view of the (a) DS $\bar{X}$ and (b) VSS $\bar{X}$ charts' operation

cumulative distribution function (cdf) and the standard normal probability density function (pdf). Likewise, $I_2^* = \left[-L+\delta\sqrt{n_1}, -L_1+\delta\sqrt{n_1}\right) \cup \left(L_1+\delta\sqrt{n_1}, L+\delta\sqrt{n_1}\right]$, $c=\sqrt{(n_1+n_2)/n_2}$, $r=\sqrt{n_1+n_2}$ and $\delta = |\mu_1-\mu_0|/\sigma_0$ denotes the magnitude of the standardised mean shift with μ_1 being the out-of-control mean. The ARL and SDRL are defined as

$$\text{ARL} = \frac{1}{1-P_a} \tag{3}$$

and

$$\text{SDRL} = \frac{\sqrt{P_a}}{1-P_a}, \tag{4}$$

respectively. Also, the ASS at each sampling time for either taking the first sample with size n_1 or the first and second samples with size $n_1 + n_2$ is

$$\text{ASS} = n_1 + n_2\left[\Phi\left(L+\delta\sqrt{n_1}\right) - \Phi\left(L_1+\delta\sqrt{n_1}\right) + \Phi\left(-L_1+\delta\sqrt{n_1}\right) - \Phi\left(-L+\delta\sqrt{n_1}\right)\right]. \tag{5}$$

The in-control process mean μ_0 and standard deviation σ_0 are usually unknown. Both parameters are estimated from an in-control Phase-I dataset which comprises m samples, each having n observations. The estimator $\hat{\mu}_0$ of μ_0 is $\hat{\mu}_0 = \sum_{k=1}^{m} \bar{X}_k / m$, where $\bar{X}_k = \sum_{j=1}^{n} X_{k,j} / n$ is the k^{th} sample mean from the Phase-I process; while the estimator $\hat{\sigma}_0$ of σ_0 is $\hat{\sigma}_0 = \sqrt{\sum_{k=1}^{m}\sum_{j=1}^{n}\left(X_{k,j}-\bar{X}_k\right)^2 / \left[m(n-1)\right]}$.

For the DS $\bar{X}$ chart with estimated process parameters, let $\hat{P}_{a1}$ and $\hat{P}_{a2}$ denote the conditional probabilities as follows (Teoh *et al.* 2014):

$$\hat{P}_{a1} = \Phi\left[U\sqrt{\frac{n_1}{mn}} + VL_1 - \delta\sqrt{n_1}\right] - \Phi\left[U\sqrt{\frac{n_1}{mn}} - VL_1 - \delta\sqrt{n_1}\right] \tag{6}$$

and

$$\hat{P}_{a2} = \int_{z \in I_2} \hat{P}_4 V \phi\left(U\sqrt{\frac{n_1}{mn}} + Vz - \delta\sqrt{n_1}\right) dz\,, \tag{7}$$

where

$$\hat{P}_4 = \Phi\left[U\sqrt{\frac{n_2}{mn}} + V\left(\frac{L_2\sqrt{n_1+n_2} - z\sqrt{n_1}}{\sqrt{n_2}}\right) - \delta\sqrt{n_2}\right] - \Phi\left[U\sqrt{\frac{n_2}{mn}} - V\left(\frac{L_2\sqrt{n_1+n_2} + z\sqrt{n_1}}{\sqrt{n_2}}\right) - \delta\sqrt{n_2}\right]. \tag{8}$$

The random variable U follows a standard normal distribution, $\hat{\mu}_0 \sim N\left[\mu_0, \sigma_0{}^2 / (mn)\right]$ and the random variable V^2 has a gamma distribution, i.e. $V^2 \sim \gamma\left[m(n-1)/2, 2/m(n-1)\right]$. Here, U and V are defined as

$$U = \left(\hat{\mu}_0 - \mu_0\right)\frac{\sqrt{mn}}{\sigma_0} \tag{9}$$

and

$$V = \frac{\hat{\sigma}_0}{\sigma_0}. \tag{10}$$

The pdfs of U and V are $f_U(u) = \phi(u)$ and $f_V(v) = 2vf_\gamma\left(v^2 \middle| m(n-1)/2, 2/m(n-1)\right)$, respectively. Note that the conditional ARL, SDRL and ASS with known process parameters are presented in Eqs. (3), (4) and (5), respectively. When the process parameters are estimated, the unconditional ARL is expressed as (Teoh *et al.* 2014):

$$\text{ARL} = \int_{-\infty}^{+\infty}\int_0^{+\infty} \frac{1}{1-\hat{P}_a} f_U(u) f_V(v)\, dv\, du\,, \tag{11}$$

where $\hat{P}_a = \hat{P}_{a1} + \hat{P}_{a2}$ is the probability that the process is in-control. The SDRL of the DS $\bar{X}$ chart with estimated process parameters is defined as

$$\text{SDRL} = \sqrt{\left[\int_{-\infty}^{+\infty}\int_0^{+\infty} \frac{1+\hat{P}_a}{(1-\hat{P}_a)^2} f_U(u) f_V(v)\, dv\, du\right] - \text{ARL}^2}\,. \tag{12}$$

Also, when the process parameters are estimated, the ASS at each sampling time is equal to

$$\text{ASS} = \int_{-\infty}^{+\infty}\int_0^{+\infty} \left(n_1 + n_2\hat{P}_2\right) f_U(u) f_V(v)\, dv\, du\,, \tag{13}$$

where the probability, $\hat{P}_2$ is as follows:

$$\hat{P}_2 = \Phi\left[U\sqrt{n_1/(mn)} - VL_1 - \delta\sqrt{n_1}\right] - \Phi\left[U\sqrt{n_1/(mn)} - VL - \delta\sqrt{n_1}\right] + \Phi\left[U\sqrt{n_1/(mn)} + VL - \delta\sqrt{n_1}\right] - \Phi\left[U\sqrt{n_1/(mn)} + VL_1 - \delta\sqrt{n_1}\right].$$

3. The VSS $\bar{X}$ Chart

Similar to the DS $\bar{X}$ chart presented in Section 2, the observations $Y'_{k,1}$, $Y'_{k,2}$, …, Y'_{k,n_k} for k = 1, 2, … are taken from the Phase-II process, where the observations in sample k are iid normal $N(\mu_0, \sigma_0^2)$ random variables. The size of the sample, which can vary between two values n_S and n_L $(n_S < n_L)$, always depends on the previous chart's statistic, $Z'_k = \left[\sqrt{n_k}\left(\bar{Y}'_k - \mu_0\right)\right]/\sigma_0$, where $\bar{Y}'_k = \sum_{j=1}^{n_k} Y'_{k,j}/n_k$ is the mean of the k^{th} subgroup or sampling time. Figure 1(b) displays a graphical view of the VSS $\bar{X}$ chart. Here, $W > 0$ and $K \geq W$ are the warning and control limits, respectively. Three conditions are considered here. If the chart's statistic, Z'_k falls within the interval $I_L = [-K, -W) \cup (W, K]$, the process is potentially shifting to an out-of-control state and a large sample size (n_L) should be taken for the next sample in order to tighten the control. If $Z'_k \in I_S = [-W, W]$, a small sample size (n_S) should be taken for the next sample. However, if Z'_k falls outside the interval $[-K, K]$, the process is out-of-control and assignable cause(s) may exist; thus, immediate actions need to be taken to remove the assignable cause(s).

According to Costa (1994), the VSS $\bar{X}$ chart can be expressed in terms of the Markov chain transition probability matrix **P** as shown below:

$$\mathbf{P} = \begin{pmatrix} \mathbf{Q} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix} = \left(\begin{array}{cc|c} P_S(n_S) & P_L(n_S) & 1 - P_S(n_S) - P_L(n_S) \\ P_S(n_L) & P_L(n_L) & 1 - P_S(n_L) - P_L(n_L) \\ \hline 0 & 0 & 1 \end{array}\right), \tag{14}$$

where **Q** is the matrix of transient probabilities and vector **r** fulfils $\mathbf{r} = \mathbf{1} - \mathbf{Q1}$ with $\mathbf{1} = (1,\ 1)^T$. Also, the probabilities $P_S(n_k)$ and $P_L(n_k)$ with $n_k = \{n_S,\ n_L\}$ are defined as

$$P_S(n_k) = \Phi\left(\delta\sqrt{n_k} + W\right) - \Phi\left(\delta\sqrt{n_k} - W\right) \tag{15}$$

and

$$P_L(n_k) = \Phi\left(\delta\sqrt{n_k} + K\right) - \Phi\left(\delta\sqrt{n_k} - K\right) + \Phi\left(\delta\sqrt{n_k} - W\right) - \Phi\left(\delta\sqrt{n_k} + W\right). \tag{16}$$

For the VSS $\bar{X}$ chart with known process parameters, the ARL and SDRL are equal to

$$\text{ARL} = \mathbf{q}^T\left(\mathbf{I}-\mathbf{Q}\right)^{-1}\mathbf{1} \tag{17}$$

and

$$\text{SDRL} = \sqrt{2\mathbf{q}^T\left(\mathbf{I}-\mathbf{Q}\right)^{-2}\mathbf{Q1} - \text{ARL}^2 + \text{ARL}}\ , \tag{18}$$

where, the vector of initial probabilities is $\mathbf{q} = \left(1,\ 0\right)^T$. The ASS of the VSS $\bar{X}$ chart is computed as

$$\text{ASS} = \left(n_S, n_L, n_S\right)\mathbf{R}^{-1}\left(1,\ 0,\ 0\right)^T, \tag{19}$$

where the matrix **R** is

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 \\ P_L(n_S) & P_L(n_L)-1 & 0 \\ 1-P_S(n_S)-P_L(n_S) & 1-P_S(n_L)-P_L(n_L) & -1 \end{pmatrix}. \tag{20}$$

When the process parameters are estimated, the estimators $\hat{\mu}_0$ and $\hat{\sigma}_0$ of the VSS $\bar{X}$ chart can be computed using the same method shown in Section 2. The conditional probabilities $\hat{P}_S\left(n_k\right)$ and $\hat{P}_L\left(n_k\right)$ derived from Castagliola *et al.* (2012) are defined as

$$\hat{P}_S\left(n_k\right) = \Phi\left(U\sqrt{\frac{n_k}{mn}} + VW - \delta\sqrt{n_k}\right) - \Phi\left(U\sqrt{\frac{n_k}{mn}} - VW - \delta\sqrt{n_k}\right) \tag{21}$$

and

$$\hat{P}_L\left(n_k\right) = \Phi\left(U\sqrt{\frac{n_k}{mn}} - VW - \delta\sqrt{n_k}\right) - \Phi\left(U\sqrt{\frac{n_k}{mn}} - VK - \delta\sqrt{n_k}\right) + \Phi\left(U\sqrt{\frac{n_k}{mn}} + VK - \delta\sqrt{n_k}\right) - \Phi\left(U\sqrt{\frac{n_k}{mn}} + VW - \delta\sqrt{n_k}\right), \tag{22}$$

where U and V are defined in Eqs. (9) and (10), respectively.

For the VSS $\bar{X}$ chart with estimated process parameters, the ARL and SDRL are defined as (Castagliola *et al.* 2012)

$$\text{ARL} = \int_{-\infty}^{+\infty}\int_{0}^{+\infty} \mathbf{q}^T\left(\mathbf{I}-\hat{\mathbf{Q}}\right)^{-1}\mathbf{1} f_U\left(u\right) f_V\left(v\right) dv du \tag{23}$$

and

$$\text{SDRL} = \sqrt{\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\left[2\mathbf{q}^T\left(\mathbf{I}-\hat{\mathbf{Q}}\right)^{-2}\hat{\mathbf{Q}}\mathbf{1} + \mathbf{q}^T\left(\mathbf{I}-\hat{\mathbf{Q}}\right)^{-1}\mathbf{1}\right] f_U\left(u\right) f_V\left(v\right) dv\, du - \text{ARL}^2}\ , \tag{24}$$

respectively, where $\hat{\mathbf{Q}}$ is the matrix $\mathbf{Q}$, which can be obtained from Eq. (14). Note that the probabilities $P_S(n_k)$ and $P_L(n_k)$ in matrix $\mathbf{Q}$ are replaced by $\hat{P}_S(n_k)$ and $\hat{P}_L(n_k)$ in matrix $\hat{\mathbf{Q}}$. The ASS for the VSS $\bar{X}$ chart with estimated process parameters is equal to

$$\text{ASS} = \int_{-\infty}^{+\infty}\int_{0}^{+\infty}(n_S, n_L, n_S)\hat{\mathbf{R}}^{-1}(1,0,0)^T f_U(u) f_V(v)\,dv\,du, \tag{25}$$

where $\hat{\mathbf{R}}$ is the matrix $\mathbf{R}$ in Eq. (20) with the conditional probabilities $\hat{P}_S(n_k)$ and $\hat{P}_L(n_k)$ replacing $P_S(n_k)$ and $P_L(n_k)$, respectively.

4. A Comparative Study on the DS $\bar{X}$ and VSS $\bar{X}$ Charts

In this section, a comparison of the ARL_0, ARL_1, SDRL_0, SDRL_1, ASS_0 and ASS_1 performances between the DS $\bar{X}$ and VSS $\bar{X}$ charts are discussed. Here, the subscripts '0' and '1' for the ARL, SDRL and ASS represent the in-control and out-of-control cases, respectively. Note that the $\text{ARL}_0 = 370.40$ and $\text{ASS}_0 = n \in \{4, 8\}$ are considered for both the DS $\bar{X}$ and VSS $\bar{X}$ charts throughout this paper.

Table 1 presents the optimal charts' parameters (n_1, n_2, L_1, L, L_2) and (n_S, n_L, W, K) for the DS $\bar{X}$ and VSS $\bar{X}$ charts, respectively, for $\delta_{\text{opt}} \in \{0.5, 1.0, 1.5\}$, $\text{ASS}_0 = n \in \{4, 8\}$, $\text{ARL}_0 = 370.40$ and $m \in \{10, 20, 40, 80, +\infty\}$. Here, δ_{opt} is the standardised mean shift, for which a quick detection is desired. Castagliola *et al.* (2012) stated that small and moderate sample sizes are commonly used in the field of industry. Therefore, $\text{ASS}_0 = n \in \{4, 8\}$ are considered throughout this paper. The optimal chart's parameters (n_1, n_2, L_1, L, L_2) for the DS $\bar{X}$ chart with known (represented with the number of the Phase-I samples, $m = +\infty$) and

Table 1: (n_1, n_2, L_1, L, L_2) combination of the optimal DS $\bar{X}$ chart and (n_S, n_L, W, K) combination of the optimal VSS $\bar{X}$ chart for both cases of known and estimated process parameters when $\text{ARL}_0 = 370.40$, $\text{ASS}_0 = n \in \{4, 8\}$, $m \in \{10, 20, 40, 80, +\infty\}$ and $\delta_{\text{opt}} \in \{0.5, 1.0, 1.5\}$

		$n = 4$		$n = 8$	
		DS $\bar{X}$	VSS $\bar{X}$	DS $\bar{X}$	VSS $\bar{X}$
δ_{opt}	m	(n_1, n_2, L_1, L, L_2)	(n_S, n_L, W, K)	(n_1, n_2, L_1, L, L_2)	(n_S, n_L, W, K)
0.5	10	(2, 13, 1.49884, 4.60072, 2.62312)	(1, 15, 1.28832, 2.84686)	(6, 9, 1.27694, 5.35805, 2.98016)	(1, 15, 0.69570, 2.98450)
	20	(2, 13, 1.46228, 5.59510, 2.69056)	(1, 15, 1.26592, 2.93325)	(6, 9, 1.24887, 5.57805, 2.98638)	(1, 15, 0.68321, 3.00263)
	40	(2, 13, 1.44409, 5.36108, 2.70426)	(1, 15, 1.25054, 2.97076)	(6, 9, 1.23477, 5.15758, 2.97810)	(2, 15, 0.73860, 3.00523)
	80	(2, 13, 1.43506, 5.28096, 2.69961)	(1, 15, 1.24202, 2.98679)	(6, 9, 1.22771, 5.09559, 2.96822)	(2, 15, 0.73429, 3.00400)

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	+∞	(2, 13, 1.42608, 5.02070, 2.67690)	(1, 15, 1.23303, 3.00000)	(6, 9, 1.22064, 5.16299, 2.95076)	(2, 15, 0.72985, 3.00000)
1.0	10	(2, 11, 1.39900, 4.00021, 2.68518)	(2, 15, 1.48574, 2.84695)	(6, 9, 1.27694, 5.35805, 2.98016)	(7, 15, 1.60121, 2.98264)
	20	(3, 10, 1.70131, 4.33676, 2.71573)	(3, 13, 1.68013, 2.93290)	(6, 9, 1.24887, 5.57805, 2.98638)	(7, 15, 1.56318, 3.00161)
	40	(3, 10, 1.67311, 5.14775, 2.73664)	(3, 15, 1.74346, 2.97038)	(6, 9, 1.23477, 5.15758, 2.97810)	(7, 15, 1.54278, 3.00485)
	80	(3, 10, 1.65895, 5.42708, 2.73675)	(3, 15, 1.72971, 2.98657)	(6, 9, 1.22771, 5.09559, 2.96822)	(7, 15, 1.53209, 3.00384)
	+∞	(3, 10, 1.64485, 5.12469, 2.72061)	(3, 15, 1.71548, 3.00000)	(6, 9, 1.22064, 5.16299, 2.95076)	(7, 15, 1.52189, 3.00000)
1.5	10	(3, 6, 1.46511, 3.69145, 2.80460)	(3, 10, 1.52839, 2.84859)	(6, 9, 1.27686, 4.62118, 2.98031)	(7, 15, 1.60121, 2.98264)
	20	(3, 6, 1.42587, 4.54789, 2.87077)	(3, 10, 1.49630, 2.93299)	(6, 9, 1.24887, 5.23773, 2.98638)	(7, 15, 1.56318, 3.00161)
	40	(3, 6, 1.40442, 5.23469, 2.89397)	(3, 10, 1.47613, 2.97035)	(6, 9, 1.23477, 5.15758, 2.97810)	(7, 15, 1.54278, 3.00485)
	80	(3, 6, 1.39369, 5.38434, 2.89853)	(3, 10, 1.46523, 2.98656)	(6, 9, 1.22771, 5.09559, 2.96822)	(7, 15, 1.53209, 3.00384)
	+∞	(3, 6, 1.38299, 5.28042, 2.89308)	(3, 10, 1.45401, 3.00000)	(6, 9, 1.22064, 5.16299, 2.95076)	(7, 15, 1.52189, 3.00000)

estimated (represented with $m \in \{10,\ 20,\ 40,\ 80\}$) process parameters are computed from the optimal design procedures proposed by Irianto and Shinozaki (1998) and Khoo *et al.* (2013b), respectively. In addition, the optimisation procedures provided by Castagliola *et al.* (2012) are used to compute the optimal chart's parameters $(n_S,\ n_L,\ W,\ K)$ for the VSS $\bar{X}$ chart with both cases of known and estimated process parameters. For example, when n = 4, m = 80 and $\delta_{\text{opt}} = 1.0$, the optimal chart's parameters ($n_1 = 3$, $n_2 = 10$, $L_1 = 1.65895$, $L = 5.42708$, $L_2 = 2.73675$) for the DS $\bar{X}$ chart with estimated process parameters produce the smallest ARL_1 value of 2.12 (Table 3) among all the possible combinations of chart's parameters; while for the VSS $\bar{X}$ chart with estimated process parameters, the smallest ARL_1 value of 3.04 (Table 3) is computed from the optimal chart's parameters ($n_S = 3$, $n_L = 15$, $W = 1.72971$, $K = 2.98657$).

Tables 2, 3 and 4 display the ARL, SDRL and ASS profiles for the DS and VSS $\bar{X}$ charts with estimated and known process parameters, for different combinations of m, n, δ and δ_{opt} when ARL_0 = 370.40. The ARL, SDRL and ASS of the DS $\bar{X}$ and VSS $\bar{X}$ charts are computed from the formulae shown in Sections 2 and 3, respectively. The optimal charts' parameters (n_1, n_2, L_1, L, L_2) and (n_S, n_L, W, K) listed in Table 1 are used to calculate the ARL, SDRL, and ASS of the DS $\bar{X}$ and VSS $\bar{X}$ charts, respectively. These ARL, SDRL and ASS values are presented in Tables 2 to 4. For instance, when n = 4, m = 20 and $\delta_{\text{opt}} = 0.5$, the optimal chart's parameters ($n_S = 1$, $n_L = 15$, $W = 1.26592$, $K = 2.93325$) (Table 1) for the VSS $\bar{X}$ chart with estimated process parameters produce ARL_1 = 28.05, SDRL_1 = 81.77 and ASS_1 = 5.67 (Table 2). With these chart's parameters, we have ARL_1 = 175.81, SDRL_1 = 452.29 and ASS_1 = 4.63 for $\delta = 0.25$; while ARL_1 = 3.73, SDRL_1 = 2.34 and ASS_1 = 4.52 for $\delta = 1.00$ (Table 2). Note that in Table 4, we only provide the ARL, SDRL and ASS values for the DS $\bar{X}$ and VSS $\bar{X}$ charts when n = 4. This is because when n = 8 and $\delta_{\text{opt}} \in \{1.0, 1.5\}$, both the DS $\bar{X}$ and VSS $\bar{X}$ charts with known and estimated process parameters have the same combination of optimal charts' parameters (Table 1). Consequently, when n = 8, the ARL, SDRL and ASS obtained in Table 3 (δ_{opt} = 1.0) will be the same as those computed in Table 4 (δ_{opt} = 1.5) ; thus, they are not presented again.

When comparing between control charts, the smallest ARL_1, SDRL and ASS values are preferred. In Tables 2 to 4, all the DS $\bar{X}$ and VSS $\bar{X}$ charts attain the same ARL_0 = 370.40 and $\text{ASS}_0 = n \in \{4, 8\}$, but different SDRL_0 values when $m \in \{10, 20, 40, 80\}$. For fixed δ_{opt}, m and n, the SDRL_0 values for the VSS $\bar{X}$ chart with estimated process parameters ($m \in \{10, 20, 40, 80\}$) are significantly higher than that of the DS $\bar{X}$ chart with estimated process parameters. This suggests that the VSS $\bar{X}$ chart's run length variability and dispersion are larger than that of the DS $\bar{X}$ chart when the process parameters are estimated. This large SDRL_0 value for the VSS $\bar{X}$ chart is unfavourable. Therefore, when the process is in-control and the process parameters are estimated, the DS $\bar{X}$ chart surpasses the VSS $\bar{X}$ chart. As m increases, the SDRL_0 values for both charts with estimated process parameters decrease and approach those of the known-process-parameter case.

With reference to Tables 2 to 4, when the process is out-of-control, it is obvious that most of the ARL_1 and SDRL_1 values for the VSS $\bar{X}$ chart are greater than that of the DS $\bar{X}$ chart. For example, when δ_{opt} = 1.0, δ = 0.25, m = 10 and n = 4, the ARL_1 and SDRL_1 values for

Table 2: ARL, SDRL and ASS of the optimal DS $\bar{X}$ and VSS $\bar{X}$ charts when $\delta_{\text{opt}} = 0.5$, ARL_0 = 370.40, $n \in \{4, 8\}$, $m \in \{10, 20, 40, 80, +\infty\}$ and $\delta \in \{0.00, 0.25, 0.50, 0.75, 1.00, 1.50, 2.00, 3.00\}$

		n = 4		n = 8	
		DS $\bar{X}$	VSS $\bar{X}$	DS $\bar{X}$	VSS $\bar{X}$
m	δ	(ARL, SDRL, ASS)	(ARL, SDRL, ASS)	(ARL, SDRL, ASS)	(ARL, SDRL, ASS)
10	0.00	(370.40, 1510.84, 4.00)	(370.40, 1767.70, 4.00)	(370.40, 771.33, 8.00)	(370.40, 768.64, 8.00)
	0.25	(172.72, 876.45, 4.30)	(212.42, 1189.43, 4.47)	(112.59, 322.88, 8.69)	(135.01, 360.72, 8.86)
	0.50	(28.27, 189.78, 5.18)	(46.22, 358.25, 5.15)	(12.58, 32.22, 10.41)	(16.56, 44.41, 9.18)
	0.75	(5.87, 19.13, 6.52)	(8.46, 49.30, 4.95)	(2.88, 3.68, 12.36)	(4.38, 4.19, 7.21)
	1.00	(2.60, 3.02, 8.13)	(3.95, 4.47, 4.39)	(1.41, 0.92, 13.79)	(2.78, 1.25, 5.63)
	1.50	(1.40, 0.81, 11.23)	(2.57, 1.29, 4.26)	(1.01, 0.13, 14.27)	(2.16, 0.64, 5.03)
	2.00	(1.11, 0.37, 12.79)	(2.05, 0.84, 4.28)	(1.00, 0.02, 11.86)	(1.91, 0.52, 4.88)
	3.00	(1.00, 0.06, 9.82)	(1.46, 0.55, 3.29)	(1.00, 0.00, 6.34)	(1.50, 0.51, 3.70)
20	0.00	(370.40, 746.32, 4.00)	(370.40, 805.22, 4.00)	(370.40, 537.49, 8.00)	(370.40, 534.01, 8.00)
	0.25	(123.36, 324.68, 4.32)	(175.81, 452.29, 4.63)	(79.88, 151.53, 8.71)	(104.79, 189.16, 9.03)
	0.50	(17.23, 38.51, 5.23)	(28.05, 81.77, 5.67)	(9.62, 13.70, 10.49)	(12.62, 17.76, 9.51)
	0.75	(4.66, 5.90, 6.63)	(6.37, 8.08, 5.29)	(2.58, 2.45, 12.47)	(4.07, 2.69, 7.24)
	1.00	(2.35, 2.04, 8.32)	(3.73, 2.34, 4.52)	(1.35, 0.75, 13.91)	(2.73, 1.12, 5.61)
	1.50	(1.36, 0.72, 11.64)	(2.56, 1.18, 4.40)	(1.01, 0.11, 14.57)	(2.16, 0.62, 5.06)
	2.00	(1.10, 0.34, 13.74)	(2.07, 0.78, 4.46)	(1.00, 0.01, 12.60)	(1.92, 0.51, 4.93)
	3.00	(1.00, 0.06, 13.39)	(1.49, 0.55, 3.49)	(1.00, 0.00, 6.44)	(1.50, 0.51, 3.75)

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40	0.00	(370.40, 528.44, 4.00)	(370.40, 548.47, 4.00)	(370.40, 445.50, 8.00)	(370.40, 444.14, 8.00)
	0.25	(91.73, 162.24, 4.32)	(150.81, 258.72, 4.71)	(63.60, 89.48, 8.73)	(89.07, 123.67, 9.12)
	0.50	(13.67, 18.59, 5.26)	(21.06, 33.91, 6.00)	(8.44, 9.56, 10.54)	(11.10, 11.84, 9.95)
	0.75	(4.21, 4.25, 6.68)	(5.77, 4.62, 5.46)	(2.44, 2.05, 12.53)	(3.76, 2.23, 7.88)
	1.00	(2.25, 1.77, 8.39)	(3.64, 2.10, 4.56)	(1.32, 0.68, 13.93)	(2.47, 0.93, 6.35)
	1.50	(1.34, 0.69, 11.73)	(2.55, 1.13, 4.47)	(1.01, 0.10, 14.24)	(1.88, 0.52, 5.58)
	2.00	(1.09, 0.32, 13.78)	(2.08, 0.75, 4.54)	(1.00, 0.01, 11.37)	(1.58, 0.52, 4.82)
	3.00	(1.00, 0.05, 13.03)	(1.51, 0.54, 3.58)	(1.00, 0.00, 6.15)	(1.11, 0.31, 2.67)
80	0.00	(370.40, 441.81, 4.00)	(370.40, 451.70, 4.00)	(370.40, 404.79, 8.00)	(370.40, 404.87, 8.00)
	0.25	(75.54, 101.96, 4.33)	(136.17, 184.09, 4.75)	(56.00, 66.21, 8.74)	(80.69, 95.43, 9.16)
	0.50	(12.20, 13.69, 5.27)	(18.27, 21.55, 6.19)	(7.87, 8.05, 10.56)	(10.43, 9.71, 10.02)
	0.75	(3.99, 3.69, 6.70)	(5.53, 3.85, 5.54)	(2.36, 1.87, 12.56)	(3.69, 2.06, 7.86)
	1.00	(2.19, 1.66, 8.43)	(3.60, 2.00, 4.58)	(1.31, 0.65, 13.95)	(2.46, 0.90, 6.34)
	1.50	(1.33, 0.67, 11.78)	(2.54, 1.10, 4.50)	(1.01, 0.10, 14.20)	(1.88, 0.51, 5.58)
	2.00	(1.09, 0.32, 13.81)	(2.08, 0.74, 4.58)	(1.00, 0.01, 11.18)	(1.58, 0.52, 4.82)
	3.00	(1.00, 0.05, 12.91)	(1.51, 0.54, 3.63)	(1.00, 0.00, 6.12)	(1.11, 0.31, 2.67)
+∞	0.00	(370.40, 369.90, 4.00)	(370.40, 369.90, 4.00)	(370.40, 369.90, 8.00)	(370.40, 369.90, 8.00)
	0.25	(60.25, 59.75, 4.33)	(120.03, 118.84, 4.77)	(48.70, 48.20, 8.74)	(72.41, 71.12, 9.19)
	0.50	(10.79, 10.28, 5.28)	(15.93, 13.93, 6.40)	(7.30, 6.78, 10.58)	(9.79, 8.03,)10.10
	0.75	(3.77, 3.23, 6.73)	(5.32, 3.33, 5.60)	(2.28, 1.71, 12.59)	(3.62, 1.91, 7.84)
	1.00	(2.14, 1.56, 8.47)	(3.56, 1.92, 4.59)	(1.29, 0.61, 13.99)	(2.45, 0.88, 6.32)
	1.50	(1.32, 0.65, 11.81)	(2.54, 1.08, 4.53)	(1.01, 0.09, 14.32)	(1.88, 0.51, 5.59)
	2.00	(1.09, 0.31, 13.77)	(2.08, 0.73, 4.62)	(1.00, 0.01, 11.44)	(1.58, 0.51, 4.83)
	3.00	(1.00, 0.05, 12.13)	(1.52, 0.54, 3.67)	(1.00, 0.00, 6.13)	(1.11, 0.31, 2.66)

the DS $\bar{X}$ chart are 176.46 and 897.42 as opposed to 215.58 and 1163.37 for the VSS $\bar{X}$ chart (Table 3). Unequivocally, the DS $\bar{X}$ chart outperforms the VSS $\bar{X}$ chart, in terms of the detection speed and variability of the run length distribution, for all ranges of shifts. However, there still exist some differences in the ASS values between the DS $\bar{X}$ and VSS $\bar{X}$ charts. For any fixed δ_{opt}, δ, *m* and *n*, the ASS_1 value of the DS $\bar{X}$ chart is smaller than that of the VSS $\bar{X}$ chart when $\delta \le 0.50$ and vice versa when $\delta \ge 0.75$. For both charts, when δ_{opt}, δ, *m* and *n* are fixed, the ARL_1 and SDRL_1 values decrease, while the ASS_1 value increases as *m* increases. Therefore, the ARL_1, SDRL_1 and ASS_1 values for the cases with estimated process parameters approach that of the control charts with known process parameters.

Table 3: ARL, SDRL and ASS of the optimal DS $\bar{X}$ and VSS $\bar{X}$ charts when $\delta_{\text{opt}} = 1.0$, $\text{ARL}_0 = 370.40$, $n \in \{4, 8\}$, $m \in \{10, 20, 40, 80, +\infty\}$ and $\delta \in \{0.00, 0.25, 0.50, 0.75, 1.00, 1.50, 2.00, 3.00\}$

m	δ	*n* = 4		*n* = 8	
		DS $\bar{X}$	VSS $\bar{X}$	DS $\bar{X}$	VSS $\bar{X}$
		(ARL, SDRL, ASS)	(ARL, SDRL, ASS)	(ARL, SDRL, ASS)	(ARL, SDRL, ASS)
10	0.00	(370.40, 1531.14, 4.00)	(370.40, 1740.74, 4.00)	(370.40, 771.33, 8.00)	(370.40, 760.49, 8.00)
	0.25	(176.46, 897.42, 4.27)	(215.58, 1163.37, 4.44)	(112.59, 322.88, 8.69)	(143.43, 362.81, 8.59)
	0.50	(30.25, 197.90, 5.04)	(49.30, 355.55, 5.28)	(12.58, 32.22, 10.41)	(19.92, 52.45, 9.71)
	0.75	(6.20, 20.89, 6.19)	(8.73, 52.60, 5.47)	(2.88, 3.68, 12.36)	(4.11, 5.59, 9.75)
	1.00	(2.58, 3.20, 7.51)	(3.52, 4.91, 5.15)	(1.41, 0.92, 13.79)	(1.99, 1.21, 8.94)

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	1.50	(1.34, 0.73, 9.77)	(2.06, 0.96, 4.81)	(1.01, 0.13, 14.27)	(1.18, 0.40, 7.61)
	2.00	(1.09, 0.33, 10.32)	(1.56, 0.63, 4.24)	(1.00, 0.02, 11.86)	(1.02, 0.12, 7.06)
	3.00	(1.00, 0.06, 6.41)	(1.10, 0.30, 2.56)	(1.00, 0.00, 6.34)	(1.00, 0.00, 7.00)
20	0.00	(370.40, 744.37, 4.00)	(370.40, 793.53, 4.00)	(370.40, 537.49, 8.00)	(370.40, 531.61, 8.00)
	0.25	(129.59, 331.86, 4.30)	(187.84, 451.02, 4.37)	(79.88, 151.53, 8.71)	(114.40, 197.06, 8.65)
	0.50	(19.20, 42.69, 5.16)	(37.63, 98.71, 5.29)	(9.62, 13.70, 10.49)	(14.99, 23.04, 9.96)
	0.75	(4.98, 6.63, 6.50)	(7.56, 12.92, 5.82)	(2.58, 2.45, 12.47)	(3.65, 3.20, 9.90)
	1.00	(2.31, 2.07, 8.06)	(3.24, 2.44, 5.57)	(1.35, 0.75, 13.91)	(1.93, 1.00, 8.99)
	1.50	(1.25, 0.58, 10.52)	(1.79, 0.78, 4.96)	(1.01, 0.11, 14.57)	(1.18, 0.39, 7.61)
	2.00	(1.05, 0.22, 10.40)	(1.32, 0.50, 4.17)	(1.00, 0.01, 12.60)	(1.01, 0.12, 7.05)
	3.00	(1.00, 0.02, 5.11)	(1.02, 0.12, 3.07)	(1.00, 0.00, 6.44)	(1.00, 0.00, 7.00)
40	0.00	(370.40, 526.72, 4.00)	(370.40, 545.94, 4.00)	(370.40, 445.50, 8.00)	(370.40, 443.54, 8.00)
	0.25	(98.55, 170.29, 4.31)	(163.95, 268.31, 4.43)	(63.60, 89.48, 8.73)	(98.20, 132.56, 8.68)
	0.50	(15.23, 20.99, 5.20)	(27.54, 45.35, 5.62)	(8.44, 9.56, 10.54)	(12.98, 15.22, 10.09)
	0.75	(4.43, 4.62, 6.59)	(6.11, 6.08, 6.18)	(2.44, 2.05, 12.53)	(3.45, 2.56, 9.96)
	1.00	(2.18, 1.73, 8.24)	(3.09, 1.88, 5.86)	(1.32, 0.68, 13.93)	(1.91, 0.92, 9.00)
	1.50	(1.23, 0.54, 11.11)	(1.81, 0.77, 5.36)	(1.01, 0.10, 14.24)	(1.17, 0.38, 7.60)
	2.00	(1.04, 0.21, 12.04)	(1.33, 0.51, 4.44)	(1.00, 0.01, 11.37)	(1.01, 0.11, 7.05)
	3.00	(1.00, 0.02, 7.77)	(1.01, 0.12, 3.09)	(1.00, 0.00, 6.15)	(1.00, 0.00, 7.00)
80	0.00	(370.40, 441.10, 4.00)	(370.40, 451.04, 4.00)	(370.40, 404.79, 8.00)	(370.40, 404.72, 8.00)
	0.25	(82.34, 109.91, 4.31)	(150.68, 197.26, 4.45)	(56.00, 66.21, 8.74)	(89.93, 105.01, 8.69)
	0.50	(13.58, 15.42, 5.22)	(23.90, 29.86, 5.74)	(7.87, 8.05, 10.56)	(12.06, 12.39, 10.15)
	0.75	(4.18, 3.95, 6.63)	(5.72, 4.69, 6.30)	(2.36, 1.87, 12.56)	(3.36, 2.32, 9.99)
	1.00	(2.12, 1.59, 8.30)	(3.04, 1.73, 5.91)	(1.31, 0.65, 13.95)	(1.89, 0.89, 9.00)
	1.50	(1.22, 0.52, 11.22)	(1.81, 0.75, 5.42)	(1.01, 0.10, 14.20)	(1.17, 0.38, 7.60)
	2.00	(1.04, 0.20, 12.34)	(1.33, 0.50, 4.48)	(1.00, 0.01, 11.18)	(1.01, 0.11, 7.05)
	3.00	(1.00, 0.01, 8.86)	(1.01, 0.12, 3.08)	(1.00, 0.00, 6.12)	(1.00, 0.00, 7.00)
$+\infty$	0.00	(370.40, 369.90, 4.00)	(370.40, 369.90, 4.00)	(370.40, 369.90, 8.00)	(370.40, 369.90, 8.00)
	0.25	(66.70, 66.19, 4.32)	(136.05, 135.26, 4.46)	(48.70, 48.20, 8.74)	(81.65, 80.93, 8.71)
	0.50	(12.00, 11.49, 5.24)	(20.66, 19.22, 5.87)	(7.30, 6.78, 10.58)	(11.20, 10.09, 10.22)
	0.75	(3.92, 3.39, 6.66)	(5.39, 3.78, 6.42)	(2.28, 1.71, 12.59)	(3.28, 2.11, 10.00)
	1.00	(2.05, 1.47, 8.35)	(2.99, 1.60, 5.96)	(1.29, 0.61, 13.99)	(1.88, 0.86, 9.00)
	1.50	(1.21, 0.50, 11.24)	(1.81, 0.73, 5.47)	(1.01, 0.09, 14.32)	(1.17, 0.38, 7.59)
	2.00	(1.04, 0.19, 12.17)	(1.33, 0.50, 4.51)	(1.00, 0.01, 11.44)	(1.01, 0.10, 7.04)
	3.00	(1.00, 0.01, 7.71)	(1.01, 0.12, 3.08)	(1.00, 0.00, 6.13)	(1.00, 0.00, 7.00)

By observing the results in Tables 2, 3 and 4, as expected, we found that the ARL and SDRL values for both charts when n = 8 are smaller than that of n = 4, for any fixed m and δ. When comparing among Tables 2 to 4, the ARL and SDRL values computed from optimal charts' parameters of $\delta_{\text{opt}} = 0.5$ (Table 2) tend to be lower for small shifts and higher for large shifts compared to those computed from optimal charts' parameters of δ_{opt} = {1.0, 1.5} (Tables 3 and 4). This indicates that the optimal charts' parameters of $\delta_{\text{opt}} = 0.5$ are more effective in detecting small shifts; while that of $\delta_{\text{opt}} = 1.5$ are more powerful for identifying large shifts.

Table 4: ARL, SDRL and ASS of the optimal DS $\bar{X}$ and VSS $\bar{X}$ charts when $\delta_{opt} = 1.5$, $ARL_0 = 370.40$, $n = 4$, $m \in \{10, 20, 40, 80, +\infty\}$ and $\delta \in \{0.00, 0.25, 0.50, 0.75, 1.00, 1.50, 2.00, 3.00\}$

		$n = 4$	
		DS $\bar{X}$	VSS $\bar{X}$
m	δ	(ARL, SDRL, ASS)	(ARL, SDRL, ASS)
10	0.00	(370.40, 1626.47, 4.00)	(370.40, 1714.11, 4.00)
	0.25	(187.83, 979.17, 4.21)	(220.94, 1123.75, 4.25)
	0.50	(37.28, 229.96, 4.77)	(58.36, 348.05, 4.83)
	0.75	(7.86, 28.25, 5.57)	(11.92, 60.23, 5.20)
	1.00	(2.85, 4.42, 6.36)	(3.86, 7.78, 5.06)
	1.50	(1.23, 0.61, 7.11)	(1.75, 0.83, 4.38)
	2.00	(1.03, 0.18, 6.24)	(1.29, 0.49, 3.77)
	3.00	(1.00, 0.01, 3.53)	(1.02, 0.12, 3.05)
20	0.00	(370.40, 759.84, 4.00)	(370.40, 792.53, 4.00)
	0.25	(144.90, 357.79, 4.22)	(190.03, 450.43, 4.30)
	0.50	(24.95, 54.99, 4.84)	(41.09, 102.13, 5.05)
	0.75	(6.27, 9.18, 5.73)	(8.76, 15.23, 5.51)
	1.00	(2.57, 2.57, 6.69)	(3.46, 2.92, 5.26)
	1.50	(1.20, 0.51, 8.01)	(1.77, 0.74, 4.49)
	2.00	(1.03, 0.16, 7.86)	(1.31, 0.49, 3.85)
	3.00	(1.00, 0.01, 4.63)	(1.02, 0.12, 3.05)
40	0.00	(370.40, 532.07, 4.00)	(370.40, 545.35, 4.00)
	0.25	(116.70, 193.76, 4.23)	(168.81, 270.46, 4.32)
	0.50	(20.42, 28.75, 4.87)	(33.42, 52.40, 5.17)
	0.75	(5.64, 6.30, 5.78)	(7.63, 8.50, 5.69)
	1.00	(2.43, 2.09, 6.77)	(3.32, 2.21, 5.35)
	1.50	(1.18, 0.47, 8.25)	(1.77, 0.71, 4.55)
	2.00	(1.02, 0.15, 8.58)	(1.32, 0.49, 3.89)
	3.00	(1.00, 0.01, 6.06)	(1.01, 0.12, 3.05)
80	0.00	(370.40, 443.80, 4.00)	(370.40, 450.88, 4.00)
	0.25	(101.29, 132.34, 4.23)	(156.40, 201.35, 4.33)
	0.50	(18.43, 21.38, 4.88)	(29.94, 36.86, 5.23)
	0.75	(5.33, 5.31, 5.80)	(7.16, 6.60, 5.78)
	1.00	(2.36, 1.89, 6.80)	(3.26, 1.98, 5.39)
	1.50	(1.17, 0.46, 8.29)	(1.78, 0.69, 4.57)
	2.00	(1.02, 0.15, 8.69)	(1.33, 0.49, 3.90)
	3.00	(1.00, 0.01, 6.42)	(1.01, 0.12, 3.05)
$+\infty$	0.00	(370.40, 369.90, 4.00)	(370.40, 369.90, 4.00)
	0.25	(85.62, 85.11, 4.23)	(142.72, 141.98, 4.34)
	0.50	(16.49, 15.99, 4.89)	(26.72, 25.49, 5.30)
	0.75	(5.00, 4.48, 5.82)	(6.73, 5.23, 5.88)
	1.00	(2.29, 1.72, 6.82)	(3.20, 1.78, 5.43)
	1.50	(1.16, 0.44, 8.31)	(1.78, 0.67, 4.59)
	2.00	(1.02, 0.14, 8.68)	(1.33, 0.49, 3.92)
	3.00	(1.00, 0.01, 6.20)	(1.01, 0.12, 3.05)

5. Conclusion

In this paper, a thorough comparison between the DS $\bar{X}$ and VSS $\bar{X}$ charts based on the performances of the ARL, SDRL and ASS are evaluated. By referring to Tables 2 to 4, the SDRL values for both the in-control and out-of-control cases of the VSS $\bar{X}$ chart are larger than that of the DS $\bar{X}$ chart. This shows that the spread of the run length distribution for the VSS $\bar{X}$ chart is higher than that of the DS $\bar{X}$ chart. Since different magnitudes of spread of the run length distributions are involved, a comparison between both charts based on the median run length and the percentiles of the run length distributions, which are more credible alternative performance measures, can be considered in future research.

The results in this paper show that the DS $\bar{X}$ chart is superior to the VSS $\bar{X}$ chart for monitoring all the process mean shifts in terms of the ARL and SDRL. However, the converse is true, in terms of the ASS when $\delta \geq 0.75$. For companies with vast production of high volumes of products, a fast out-of-control detection speed will be of main interest as such companies do not face problems involving large sample sizes. Such companies may prefer applying the DS $\bar{X}$ chart to monitor their production processes as the DS $\bar{X}$ chart detects changes in the process mean faster than the VSS $\bar{X}$ chart. If the sample size is a major constraint, we recommend applying the VSS $\bar{X}$ chart.

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