Journal of Quality Measurement and Analysis
Jurnal Pengukuran Kualiti dan Analisis

JQMA 10(2) 2014, 41-50

ON CERTAIN SUBCLASS OF *p*-VALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

(Berkenaan Subkelas Fungsi *p*-Valen Tertentu Berpekali Positif)

ANAS ALJARAH & MASLINA DARUS

ABSTRACT

In this article, a certain differential operator $S^k_{\alpha,\beta,\lambda,\delta,p} f(z)$ and a subclass $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$ for analytic functions with positive coefficients of the form $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ are introduced. Some properties such as the coefficients inequalities, distortion theorem, radii of starlikeness and convexity, closure theorems and extreme points are given. A class preserving an integral operator is also stated.

Keywords: analytic function; starlike functions; convex functions; integral operator

ABSTRAK

Dalam makalah ini, pengoperasi pembeza tertentu $S^k_{\alpha,\beta,\lambda,\delta,p} f(z)$ dan subkelas $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$ bagi fungsi analisis berpekali positif dalam bentuk $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ diperkenalkan. Beberapa sifat seperti ketaksamaan pekali, teorem erotan, jejari kebakbintangan dan kecembungan, teorem tutupan dan titik ekstrem diberi. Kelas mengekalkan pengoperasi kamiran juga dinyatakan.

Kata kunci: fungsi analisis; fungsi bakbintang; fungsi cembung; pengoperasi kamiran

1. Introduction and Preliminaries

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (z \in U, p \in \mathbb{N} := \{1,2,3,...\}), \tag{1.1}$$

which are analytic and *p*-valent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in $A(p)$ is said to be *p*-valent starlike of order γ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \qquad (z \in U), \tag{1.2}$$

for some $\gamma (0 \le \gamma \le p)$. We denote by $S^*(\gamma)$ the class of all *p*-valent starlike functions of order γ.

Similarly, a function $f(z)$ in $A(p)$ is said to be *p*-valent convex of order γ if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f(z)}\right\} > \gamma \qquad (z \in U), \tag{1.3}$$

for some $\gamma (0 \le \gamma < p)$. We denote by $C(\gamma)$ the class of all *p*-valent convex of order γ.

For the function $f \in A(p)$, we define the following new differential operator:

$$S^0{}_{\alpha,\beta,\lambda,\delta,p} f(z) = f(z),$$

$$S^1{}_{\alpha,\beta,\lambda,\delta,p} f(z) = [1 - p(\lambda-\delta)(\beta-\alpha)]f(z) + [(\lambda-\delta)(\beta-\alpha)]zf^{'}(z),,$$

and

$$S^k_{\alpha,\beta,\lambda,\delta,p} f(z) = S\left(S^{k-1}_{\alpha,\beta,\lambda,\delta,p} f(z)\right)$$
$$= z^p + \sum_{n=p+1}^{\infty} \left((\lambda-\delta)(\beta-\alpha)(n-p)+1\right)^k a_n z^n, \tag{1.4}$$

for $f \in A(p), \alpha, \beta, \delta, \lambda \geq 0, \lambda > \delta, \beta > \alpha, p \in \mathbb{N}, k = 1,2,3,...$ and $n \in \mathbb{N} = \mathbb{N} \cup \{0\}$.

It is easily verified from (1.4) that

$$(\lambda-\delta)(\beta-\alpha)z(S^k_{\alpha,\beta,\lambda,\delta,p} f(z))^{'} = S^{k+1}_{\alpha,\beta,\lambda,\delta,p} f(z) - (1 - p(\lambda-\delta)(\beta-\alpha))S^k_{\alpha,\beta,\lambda,\delta,p} f(z). \tag{1.5}$$

Remark 1: When $p = 1$, we have the operator introduced and studied by Ramadan and Darus (2011).

Making use of the operator $S^k_{\alpha,\beta,\lambda,\delta,p} f(z)$ we say that a function $f(z) \in A(p)$ is in the class $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$ if it satisfies the following inequality (Aouf 2012):

$$\Re\left\{z^p S^k f(z) - \alpha z^{p+1}\left(S^k f(z)\right)^{'}\right\} > \beta, \tag{1.6}$$

for some $\alpha(\alpha > 0), \beta(0 \leq \beta < p), p \in \mathbb{N}, n \in \mathbb{N}_0$ and for all $z \in U$. We note that (Altintas *et al.* 1994),

$$M^0(1,\alpha,\beta) = M(1,\alpha,\beta). \tag{1.7}$$

In Uraleggadi and Ganigi (1986), they defined the class $H^*_p(\alpha)$ as follows:

$$\left\{f(z) \in A(p) : \Re\left\{-z^{p+1} f'(z)\right\} > \alpha, 0 \leq \alpha < p, z \in U\right\}. \tag{1.8}$$

Some other subclasses of the class $A(p)$ were studied (for example) by Cho *et al.* (1987, 1989), Liu (2000), Liu and Srivastava (2001; 2004), Joshi *et al.* (1995), Raina and Srivastava (2006), Aouf and Shammaky (2005).

In this paper, coefficient inequalities, distortion theorem, radii of starlikeness and convexity, closure theorems, extreme points and the class preserving integral operators of the form

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt \text{ for all } c > -p,$$

for the class $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$ are considered.

2. Coefficient Inequalities

In this section, we give a necessary and sufficient condition for a function $f(z)$ given by (1.1) to be in the class $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$.

Theorem 2.1. *A function* $f(z)$ *given by* (1.1) *is in* $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$*, where* $\alpha>0$*,* $\beta,\delta,\lambda\geq 0,$*,* $p\in\mathbb{N}$ *and* $n\in\mathbb{N}_0$ *if and only if*

$$\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)a_n\leq(1-\alpha p-\beta). \tag{2.1}$$

Proof. Suppose that $f(z)\in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$. Then we find from (1.6) that

$$\Re\left\{\begin{array}{l} z^p\left[z^p+\displaystyle\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k a_n z^n\right] \\ -\alpha z^{p+1}\left[n\displaystyle\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p)+1)^k a_n z^{n-1}\right]\end{array}\right\}>\beta$$

$$\Re\left\{(1-\alpha p)z^{2p}-\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)a_n z^{n+p}\right\}>\beta$$

$$(\alpha>0,\ \beta,\delta,\lambda\geq 0,\ p\in\mathbb{N}\ \textit{and}\ n\in\mathbb{N}_0).$$

If we choose $|z|=r<1$, then we have

$$1-\alpha p-\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)a_n\geq\beta$$

for $\alpha>0$, $\beta,\delta,\lambda\geq 0$, $p\in\mathbb{N}$ *and* $n\in\mathbb{N}_0$, which is equivalent to (2.1).

Conversely, let us suppose that the equivalent (2.1) holds true. Then we have

$$\begin{aligned} &\left|z^p S^n f(z)-\alpha z^{p+1}(S^n f(z))-(1-\alpha p)z^{2p}\right| \\ &\quad=\left|-\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)a_n z^{n+p}\right| \\ &\quad\leq\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)a_n\left|z^{n+p}\right| \\ &\quad\leq 1-\alpha p-\beta\end{aligned}$$

for $\alpha>0$, $\beta,\delta,\lambda\geq 0$, $p\in\mathbb{N}$ and $n\in\mathbb{N}_0$, which implies that $f(z)\in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$. This gives the required condition. Hence the theorem follows. □

Corollary 2.2. *Let the function* $f(z)$ *be defined by* (1.1). *If* $f\in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$. *Then*

$$a_n \le \frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)} z^n \qquad (k \ge p; p \in \mathbb{N}; n \in \mathbb{N}_0). \qquad (2.2)$$

The equality in (2.2) *is attained for the functions* $f(z)$ *given by*

$$f(z) = z^p + \frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)} z^n \qquad (k \ge p; p \in \mathbb{N}; n \in \mathbb{N}_0). \qquad (2.3)$$

3. Distortion Theorem

A distortion property for functions in the class $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$ is contained in the following theorem.

Theorem 3.1. *Let the functions* $f(z)$ *given by* (1.1) *be in the class* $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$, *where* $\alpha > 0$, $\beta,\delta,\lambda \ge 0$, $p \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$. *Then for* $|z| = r < 1$, *we have*

$$r^p - \frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)} r^{1+p}$$
$$\le |f(z)| \le r^p + \frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)} r^{1+p}$$

and

$$p - \frac{(1-\alpha p-\beta)(1+p)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)} r^p$$
$$\le |f'(z)| \le p + \frac{(1-\alpha p-\beta)(1+p)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)} r^p.$$

Proof. Since $f \in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$, from Theorem 2.1 readily yields the inequality

$$\sum_{n=p+1}^{\infty} a_n \le \frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)}.$$

Thus, for $|z| = r < 1$, and making use of (3.1) we have

$$|f(z)| \le |z|^p + \sum_{n=p+1}^{\infty} a_n |z|^n \le r^p + r^{p+1} \sum_{n=p+1}^{\infty} a_n$$
$$\le r^p + \frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)} r^{p+1}$$

and

$$|f(z)| \ge |z|^p - \sum_{n=p+1}^{\infty} a_n |z|^n \le r^p - r^{p+1} \sum_{n=p+1}^{\infty} a_n$$
$$\ge r^p - \frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)} r^{p+1}.$$

Also from Theorem 2.1, it follows that

$$\frac{((\lambda-\delta)(\beta-\alpha)(n-p))^{k}(\alpha n-1)}{1+p}\sum_{n=p+1}^{\infty} na_n$$

$$\leq \sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p))^{k}(\alpha n-1)a_n \leq (1-\alpha p-\beta). \tag{3.2}$$

Hence

$$\left|f^{'}(z)\right| \leq p|z|^{p-1}+\sum_{n=p+1}^{\infty} na_n|z|^{n-1} \leq pr^{p-1}+r^{p}\sum_{n=p+1}^{\infty} na_n$$

$$\leq p+\frac{(1-\alpha p-\beta)(1+p)}{((\lambda-\delta)(\beta-\alpha)(n-p))^{k}(\alpha n-1)}r^{p}$$

and

$$\left|f^{'}(z)\right| \geq p|z|^{p-1}-\sum_{n=p+1}^{\infty} na_n|z|^{n-1} \geq pr^{p-1}-r^{p}\sum_{n=p+1}^{\infty} na_n$$

$$\geq p-\frac{(1-\alpha p-\beta)(1+p)}{((\lambda-\delta)(\beta-\alpha)(n-p))^{k}(\alpha n-1)}r^{p}.$$

This completes the proof of Theorem 3.1. □

4. Radii of Starlikeness and Convexity

In this section, we determine the radii of starlikeness and convexity for functions in the class $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$. The result is given as follows.

Theorem 4.1. *Let the function* $f(z)$ *defined by* (1.1) *is in the class* $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$, *where* $\alpha>0$, $\beta,\delta,\lambda\geq 0$, $p\in\mathbb{N}$ *and* $n\in\mathbb{N}_0$. *Then* $f(z)$ *is starlike of order* $\delta(0\leq\delta<1)$ *in* $|z|<r_1$, *where*

$$r_1=\inf_{n\geq p+1}\left\{\frac{(p-\delta)(p-\delta)((\lambda-\delta)(\beta-\alpha)(n-p))^{k}(\alpha n-1)}{(1-n-p+\delta)(1-\alpha p-\beta)}\right\}^{\frac{1}{n}} \qquad (n\geq p+1). \tag{4.1}$$

The result is sharp for the function $f(z)$ *given by* (2.3).

Proof. It suffices to prove that

$$\left|\frac{zf^{'}(z)}{f(z)}-p\right|\leq p-\delta, \tag{4.2}$$

for $|z|<r_1$. We have

$$\left|\frac{zf'(z)}{f(z)}-p\right|=\left|\frac{-\sum_{n=p+1}^{\infty}(1-n)a_nz^n}{z^p+\sum_{n=p+1}^{\infty}a_nz^n}\right|\le\frac{\sum_{n=p+1}^{\infty}(1-n)a_nz^n}{1+\sum_{n=p+1}^{\infty}a_n|z|^n}. \tag{4.3}$$

Hence (4.3) holds true if

$$\sum_{n=p+1}^{\infty}(1-n)a_n|z|^n\le(p-\delta)\left(1+\sum_{n=p+1}^{\infty}a_n|z|^n\right)$$

or

$$\sum_{n=p+1}^{\infty}\frac{(1-n-p+\delta)}{(p-\delta)}a_n|z|^n\le 1 \tag{4.4}$$

With the aid of (2.1), (4.4) is true if

$$\frac{(1-n-p+\delta)}{(p-\delta)}|z|^n\le\frac{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)}{(1-\alpha p-\beta)}. \tag{4.5}$$

Solving (4.5) for $|z|^n$, we obtain

$$|z|\le\left\{\frac{(p-\delta)(p-\delta)((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)}{(1-n-p+\delta)(1-\alpha p-\beta)}\right\}^{\frac{1}{n}}\qquad(n\ge p+1).$$

This completes the proof of Theorem 4.1. □

Theorem 4.2. *Let the function* $f(z)$ *defined by* (1.1) *is in the class* $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$, *where* $\alpha>0$, $\beta,\delta,\lambda\ge 0$, $p\in\mathbb{N}$ *and* $n\in\mathbb{N}_0$. *Then* $f(z)$ *is convex of order* $\delta(0\le\delta<1)$ *in* $|z|<r_2$, *where*

$$r_2=\inf_{n\ge p+1}\left\{\frac{p(1-\delta)((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)}{n(1-n-p+\delta)(1-\alpha p-\beta)}\right\}^{\frac{1}{n-1}}\qquad(n\ge p+1). \tag{4.6}$$

The result is sharp for the functions $f(z)$ *given by* (2.3).

Proof. By using the same technique employed in the proof of Theorem 4.1, we can show that

$$\left|\frac{zf''(z)}{f'(z)}\right|\le p-\delta, \tag{4.7}$$

for $|z|<r_2$, with the aid of Theorem 2.1. Thus, we have the assertion of Theorem 4.2. □

5. Closure Theorems

Let the functions $f_j(z)$, $j = 1,2,...,l$ be defined by

$$f_j(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,j} z^n \qquad (a_{n,j} \geq 0), \tag{5.1}$$

for $z \in U$.

In this section, we prove the closure theorem for the class $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$. The results are given in the following theorems.

Theorem 5.1. *Let the functions* $f_j(z)$ *defined by* (5.1) *be in the class* $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$, *where* $\alpha > 0$, $\beta,\delta,\lambda \geq 0$, $p \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$ *for every* $j = 1,2,...,l$. *Then the function* $G(z)$ *defined by*

$$G(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \qquad (b_n \geq 0) \tag{5.2}$$

is a member of the class $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$, *where*

$$b_n = \frac{1}{l}\sum_{j=1}^{l} a_{n,j} \qquad (n \geq p+1).$$

Proof. Since $f_j(z) \in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$, it follows from Theorem 2.1 that

$$\sum_{n=p+1}^{\infty} ((\lambda-\delta)(\beta-\alpha)(n-p))^k (\alpha n - 1) a_n \leq (1 - \alpha p - \beta).$$

for every $j = 1,2,...,l$. Hence,

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} ((\lambda-\delta)(\beta-\alpha)(n-p))^k (\alpha n - 1) b_n \\
&= \sum_{n=p+1}^{\infty} ((\lambda-\delta)(\beta-\alpha)(n-p))^k (\alpha n - 1) \left\{\frac{1}{l}\sum_{j=1}^{l} a_{n,j}\right\} \\
&= \frac{1}{l}\sum_{j=1}^{l}\left(\sum_{n=p+1}^{\infty} ((\lambda-\delta)(\beta-\alpha)(n-p))^k (\alpha n - 1) a_{n,j}\right) \\
&\leq \frac{1}{l}\sum_{j=1}^{l} (1 - \alpha p - \beta) = (1 - \alpha p - \beta),
\end{aligned}$$

which (in view of Theorem 2.1) implies that $G(z) \in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$. □

Theorem 5.2. *The class* $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$ *is closed under convex linear combination, where* $\alpha > 0$, $\beta,\delta,\lambda \geq 0$, $p \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$.

Proof.

$$f_j(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,j} z^n \qquad (a_{n,j} \geq 0; j = 1,2; p \in \mathbb{N}) \tag{5.3}$$

be in the class $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$. It is sufficient to show that the function *h(z)* defined by

$$h(z) = t f_1(z) + (1-t) f_2(z) \qquad (0 \le t \le 1) \tag{5.4}$$

is also in the class $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$. Since for $0 \le t \le 1$,

$$h(z) = z^p + \sum_{n=p+1}^{\infty} \left[t a_{n,1} + (1-t) a_{n,2} \right] z^n \qquad (0 \le t \le 1), \tag{5.5}$$

we observe that

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} ((\lambda-\delta)(\beta-\alpha)(n-p))^k (\alpha n - 1)\left[t a_{n,1} + (1-t) a_{n,2} \right] \\
&\quad = t \sum_{n=p+1}^{\infty} ((\lambda-\delta)(\beta-\alpha)(n-p))^k (\alpha n - 1) a_{n,1} \\
&\quad + (1-t) \sum_{n=p+1}^{\infty} ((\lambda-\delta)(\beta-\alpha)(n-p))^k (\alpha n - 1) a_{n,2} \\
&\quad \le t(1-\alpha p-\beta) + (1-t)(1-\alpha p-\beta) = (1-\alpha p-\beta).
\end{aligned}$$

Hence $h(z) \in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$. This completes the proof of Theorem 5.2. □

6. Integral Operators

In this section, we consider the integral transforms of functions in the class $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$.

Theorem 6.1. *If the function* $f(z)$ *given by* (1.1) *is in the class* $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$, *where* $\alpha > 0$, $\beta,\delta,\lambda \ge 0$, $p \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$ *and let c be a real number such that* $c > -p$, *then the function* $F(z)$ *defined by*

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{6.1}$$

is also belongs to the class $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$.

Proof. From (6.1), it follows that $F(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$, where $b_n = \left(\frac{c+p}{n+c} \right) a_n$. Therefore

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} ((\lambda-\delta)(\beta-\alpha)(n-p))^k (\alpha n - 1) b_n \\
&\quad = \sum_{n=p+1}^{\infty} ((\lambda-\delta)(\beta-\alpha)(n-p))^k (\alpha n - 1) \left(\frac{c+p}{n+c} \right) a_n \\
&\quad \le \sum_{n=p+1}^{\infty} ((\lambda-\delta)(\beta-\alpha)(n-p))^k (\alpha n - 1) a_n \le (1-\alpha p-\beta),
\end{aligned}$$

since $f(z) \in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$. Hence by Theorem 2.1, $F(z) \in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$. □

7. Extreme Points

Now, we determine extreme points for functions in the class $S_{n,p}(\alpha,\beta,\delta,\lambda,p)$.

Theorem 7.1. *Let* $f_p(z)=z^p$ *and*

$$f_n=z^p+\frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)}z^n.$$

Then $f\in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$ *if and only if it can be expressed in the form*

$$f(z)=\xi_p f_p(z)+\sum_{n=p+1}^{\infty}\xi_n f_n(z),\quad z\in U\text{, where }\xi_n\ge 0\text{ and }\xi_p=1-\sum_{n=p+1}^{\infty}\xi_n.$$

Proof. Suppose that

$$f(z)=\xi_p f_p(z)+\sum_{n=p+1}^{\infty}\xi_n f_n(z)$$

$$f(z)=\left[1-\sum_{n=p+1}^{\infty}\xi_n\right]z^p+\sum_{n=p+1}^{\infty}\xi_n\left\{z^p+\frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)}z^n\right\}$$

$$=z^p+\sum_{n=p+1}^{\infty}\xi_n\frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)}z^n.$$

Then from Theorem 2.1 we have

$$\sum_{n=p+1}^{\infty}((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)\frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)}\xi_n\le(1-\alpha p-\beta).$$

Hence $f\in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$.

Conversely, let $f\in S_{n,p}(\alpha,\beta,\delta,\lambda,p)$. Using Corollary 2.2, we have

$$a_n\le\frac{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)}{(1-\alpha p-\beta)}.$$

We may set

$$\xi_n=\frac{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)}{(1-\alpha p-\beta)}a_n$$

and letting $\xi_p=1-\sum_{n=p+1}^{\infty}\xi_n$, we have

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^p = z^p - \sum_{n=p+1}^{\infty} \xi_n z^p + \sum_{n=p+1}^{\infty} \xi_n z^p + \sum_{n=p+1}^{\infty} \xi_n \frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)} z^n$$

$$= \left[1 - \sum_{n=p+1}^{\infty} \xi_n\right] z^p + \sum_{n=p+1}^{\infty} \xi_n \left[z^p + \frac{(1-\alpha p-\beta)}{((\lambda-\delta)(\beta-\alpha)(n-p))^k(\alpha n-1)} z^n \right]$$

$$= \xi_p f_p(z) + \sum_{n=p+1}^{\infty} \xi_n f_n(z).$$

This completes the proof of the theorem. □

References

Altintas O., Irmark H. & Srivastava H. M. 1994. Family of meromorphically univalent functions with positive coefficients. *PanAmer J. Math.* **5**: 75-81.

Ramadan S. F. & Darus M. 2011. On the Fekete-Szego inequality for a class of analytic functions defined by using generalized differential operator. *Acta Universitatis Apulensis* **26**: 167-178.

Uralegaddi B. A. & Ganigi M. D. 1986. Meromorphic multivalent functions with positive coefficients. *Nepali Math. Sci. Rep.* **11**(2): 95-102.

Cho N. E., Lee S. H. & Owa S. 1987. A class of meromorphic univalent functions with positive coefficients. *Kobe J. Math.* **4**: 43-50.

Cho N. E., Owa S., Lee S. H. & Altintas O. 1989. Generalization class of certain meromorphic univalent functions with positive coefficients. *Kyungpook J. Math.* **29**: 133-139.

Jinlin L. 2000. Properties of some families of meromorphic p-valent functions. *Mathematica Japonicae* **52**(3): 425-434.

Liu J. L. & Srivastava H. M. 2001. A linear operator and associated families of mero-morphically multivalent functions. *J. Math. Anal. Appl* **259**: 566-581.

Liu J. L. & Srivastava H. M. 2004. Classes of meromorphically multivalent functions associated with the generalized hypergeometric function. *Mathematical and Computer Modelling* **39**(1): 21-34.

Joshi S. B., Kulkarni S. R. & Srivastava H. M. 1995. Certain classes of meromorphic functions with positive and missing coefficients. *Journal of Mathematical Analysis and Applications* **193**(1): 1-14.

Raina R. K. & Srivastava H. M. 2006. A new class of meromorphically multivalent functions with applications to generalized hypergeometric functions. *Mathematical and Computer Modelling* **43**(3): 350-356.

Aouf M. K. & Shammaky A. E. 2005. A certain subclass of meromorphically $p-$valent convex functions with negative coefficients. *J. Approx. Theory and Appl.* **1**(2): 157-177.

Aouf M. K. 2012. A family of meromorphically *p*-valent functions with positive coefficients. *General Mathematics* **20**(1): 25-37.

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi
Selangor DE, MALAYSIA
*E-mail: anasjrah@yahoo.com, maslina@ukm.edu.my**

* Corresponding author