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NOTES ON CONJUGACIES AND RENORMALISATIONS OF CIRCLE DIFFEOMORPHISMS WITH BREAKS

(Catatan mengenai Kekonjugatan dan Penormalan Semula bagi Difeomorfisma Bulatan dengan Titik Putus-putus)

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ABSTRACT

Let f be an orientation-preserving circle diffeomorphism with irrational "rotation number" of bounded type and finite number of break points, that is, the derivative f' has discontinuities of first kind at these points. Suppose f' satisfies a certain Zygmund condition which be dependent on parameter $\gamma > 0$ on each continuity intervals. We prove that the Rauzy-Veech renormalisations of f are approximated by Mobius transformations in C^1-norm if $\gamma \in (0,1]$ and in C^2-norm if $\gamma \in (1,\infty)$. In particular, we show that if f has zero mean nonlinearity, renormalisation of such maps approximated by piecewise affine interval exchange maps. Further, we consider two circle homeomorphisms with the same irrational "rotation number" of bounded type and finite number of break points. We prove that if they are not break equivalent then the conjugating map between these two maps is singular.

Keywords: conjugacy; circle diffeomorphism; break point; renormalisation; interval exchange transformation; Mobius transformation; Rauzy-Veech induction

ABSTRAK

Andaikan f suatu difeomorfisma bulatan mengawet orientasi dengan "nombor putaran" tak nisbah jenis terbatas dan dengan bilangan titik putus yang terhingga, iaitu terbitan f' mempunyai ketakselanjaran jenis pertama pada titik-titik tersebut. Andaikan juga f' memenuhi syarat Zygmund yang bersandar kepada parameter $\gamma > 0$ atas setiap selang keselanjaran. Dibuktikan bahawa penormalan semula Rauzy-Veech f dihampirkan oleh penjelmaan Mobius dalam norma-C^1 jika $\gamma \in (0,1]$ dan dihampirkan dalam norma-C^2 jika $\gamma \in (1,\infty)$. Khususnya, ditunjukkan bahawa jika f mempunyai penormalan semula ketaklinearan min sifar, penghampiran berkenaan merupakan pemetaan pertukaran linear selang afin cebis demi cebis. Tambahan kami turut mempertimbangkan dua homeomorfisma bulatan yang mempunyai "nombor putaran" tak nisbah yang sama jenis terbatas dan bilangan titik putus yang terhingga. Kami buktikan jika kedua-dua pemetaan tersebut tidak setara terputus, maka pemetaan berkonjugat di antara mereka adalah singular.

Kata kunci: konjugasi; difeomorfisma bulatan; titik putus; pennormalan semula; penjelmaan pertukaran selang; penjelmaan Mobius; aruhan Rauzy-Veech

1. Introduction

In this work we announce our new results regarding conjugacies and renormalisations of circle diffeomorphisms with several break points in short form. The problems on conjugacies and renormalisations of circle diffeomorphisms are the most actual problems in the theory of circle dynamics. Nowadays these problems have been intensively study by many researchers. The origin of the problem of singularity of conjugacy of piecewise linear circle homeomorphisms with two break points goes back to Herman in 1979. Since then the generalisation of Herman's result for the general case, that is homeomorphisms with $n \geq 3$ break points and trivial product jump has been opened. We have solved this problem under a certain condition. Our proof is based on to analyse renormalisations of Sinai and Knanin (1989), Mackay (1988) and Stark (1988).

Poincare in 1885 noticed that the orbit structure of orientation-preserving differomorphism f is determined by some irrational mod 1, called the rotation number of f and denoted by $\rho = \rho(f)$, in the following sense: for any $x \in S^1$ the mapping $f^j(x) \to j\rho$ is orientation-preserving. Denjoy (1932) proved that if f is the orientation-preserving C^1 -diffeomorphism of the circle with irrational rotation number ρ and $\log f'$ has bounded variation then, the orbit $\{f^j(x)\}_{j\in\mathbb{Z}}$ is dense and the mapping $f^j(x) \to j\rho$ mod 1 can therefore be extended by continuity to a homeomorphism h of S^1 which conjugates f to the linear rotation $f_\rho : x \to x + \rho$ mod 1. The problem of smoothness of the conjugacy of smooth diffeomorphisms has come to be very well understood by authors (Herman 1979; Yoccoz 1984; Khanin & Sinai 1987, 1989; Katznelson & Ornstein 1989). They have shown that if f is C^3 or $C^{2+\alpha}$ and ρ satisfies certain Diophantine condition then the conjugacy will be at least C^1. A natural generalisation of diffeomorphism of the circle are diffeomorphisms with breaks, those are, circle diffeomorphisms which are smooth everywhere with the exception of finitely many points at which their derivative has discontinuities of the first kind. Circle diffeomorphisms with breaks were introduced by Khanin and Vul (1990; 1991) at the beginning of 90's. They proved that the renormalisations of $C^{2+\alpha}$ diffeomorphisms converge exponentially to a two-dimensional space of the Mobius transformations. Recently Cunha and Smania (2013) studied Rauzy-Veech renormalisations of $C^{2+\alpha}$ circle diffeomorphisms with several break points. The main idea of this work is to consider piecewise-smooth circle homeomorphisms as generalised interval exchange transformations. They have proved that Rauzy-Veech renormalisations of $C^{2+\alpha}$ generalised interval exchange maps satisfying a certain combinatorial conditions are approximated by Mobius transformations in C^2 -norm. In this work we have generalised their result to a wider class of circle diffeomorphisms the so-called *Zygmund class*. Further, we consider two circle homeomorphisms with the same irrational rotation number of bounded type and finite number of break points. We study these maps as the generalised interval exchange maps. We prove that if two such circle homeomorphisms are not break equivalent then the conjugating map between them is singular. In particular, if one of them is pure rotation then the invariant measure of second one is singular with respect to Lebesgue measure.

2. Generalised interval exchange maps

Let I be an open bounded interval. A generalised interval exchange map (g.i.e.m) f on I is defined by the following data. Let A be an alphabet with $d \geq 2$ symbols. Consider a partition $(\text{mod}\, 0)$ of I into d open subintervals indexed by $I = \cup I_\alpha$. The map f is defined on $\cup I_\alpha$

and its restriction to each I_α is an orientation preserving homeomorphism onto the $f(I_\alpha)$. Let $r>1$ be an integer. The g.i.e.m f is of class C^r if the restriction of f to each I_α extends to a C^r-diffeomorphism from the closure of I_α onto of closure of $f(I_\alpha)$. The points $u_1<...<u_{d-1}$ separating the I_α are called the singularities (break points) of f.

3. Rauzy-Vech Induction

A pair $\pi=(\pi_0,\pi_1)$ of bijections $\pi_\varepsilon : A\to\{1,...,d\}$, $\varepsilon\in\{0,1\}$ describing the ordering of the subintervals I_α before and after the map is iterated. For each $\varepsilon\in\{0,1\}$, define $\alpha(\varepsilon)=\pi_\varepsilon^{-1}(d)$. If $\left|I_{\alpha(0)}\right|\neq\left|f(I_{\alpha(1)})\right|$ we say that f is Rauzy-Veech renormalisable (or simply renormalisable). If $\left|I_{\alpha(0)}\right|>\left|f(I_{\alpha(1)})\right|$ we say that the letter $\alpha(0)$ is the winner and the letter $\alpha(1)$ is the loser, we say that f is type 0 renormalisable and we can define a map $R(f)$ as the first return map of f to the interval $I^1=I\setminus f(I_{\alpha(1)})$. Otherwise $\left|I_{\alpha(0)}\right|<\left|f(I_{\alpha(1)})\right|$, the letter $\alpha(1)$ is the winner and the letter $\alpha(0)$ is the loser, we say that f is type 1 renormalisable and we can define a map $R(f)$ as the first return map of f to the interval $I^1=I\setminus f(I_{\alpha(0)})$. We want to see $R(f)$ as a g.i.e.m. To this end we need to associate to this map an A-indexed partition of its domain. We do this in the following way. The subintervals of the A-indexed partition P^1 of I^1 are denoted by I^1_α. If f has type 0, then $I^1_\alpha=I_\alpha$ when $\alpha\neq\alpha(0)$ and $I^1_{\alpha(0)}=I_{\alpha(0)}\setminus f(I_{\alpha(1)})$. If f has type 1, $I^1_\alpha=I_\alpha$ when $\alpha\neq\alpha(1),\alpha(0)$ and $I^1_{\alpha(1)}=f^{-1}(f(I_{\alpha(1)})\setminus I_{\alpha(0)}),I^1_{\alpha(0)}=I_{\alpha(1)}\setminus I^1_{\alpha(1)}$. It is easy to see that both cases (type 0 and 1) we have

$$R(f)(x)=\begin{cases} f^2(x), & \text{if } \; x\in I^1_{\alpha(1-\varepsilon)},\\ f(x), & \text{otherwise.}\end{cases}$$

And $(R(f),A,P^1)$ is a g.i.e.m, called the Rauzy-Veech renormalisation (or simply renormalisation) of f. A g.i.e.m. is infinitely renormalisable if $R^n(f)$ is well defined, for every $n\in\mathbb{N}$. For every interval of the form $J=[a,b)$ we denote $\partial J=\{a\}$. We say that a g.i.e.m. f has no connection if $f^m(\partial I_\alpha)\neq\partial I_\beta$ for all $m\geq 1$, $\alpha,\beta\in A$ with $\pi_0(\beta)\neq 1$. Let ε_n be the type of the n-th renormalisation, $\alpha_n(\varepsilon_n)$ be the winner and $\alpha_n(1-\varepsilon_n)$ be the loser of the n-th renormalisation. We say that infinitely renormalisable g.i.e.m. f has k-bounded combinatorics (i.e., “rotation number” is bounded type) if for each n and $\beta,\gamma\in A$ there exist

$n_1, p \geq 0$ with $|n - n_1| < k$ and $|n - n_1 - p| < k$ such that $\alpha_{n_1}(\varepsilon_{n_1}) = \beta,\ \alpha_{n_1+p}(1 - \varepsilon_{n_1+p}) = \gamma$ and $\alpha_{n_1+i}(1 - \varepsilon_{n_1+p}) = \alpha_{n_1+i+1}(\varepsilon_{n_1+i})$ for every $0 \leq i < p$. We say that a g.i.e.m. $f : I \to I$ has genus one if f has at most two discontinuities. Note that every g.i.e.m. with either two or three intervals has genus one. If f is renormalisable and has genus one, it is easy to see that $R(f)$ has genus one. Note that an acceleration of Rauzy-Veech renormalisation is the Mackay-Stark renormalisation and an acceleration of Mackay-Stark renormalisation is the Sinai-Knanin renormalisation.

4. Zygmund Class

To formulate our results we have to define a new class. For this we consider the function $Z^{\gamma} : [0,1) \to (0, +\infty)$ such that $Z^{\gamma}(0) = 0$ and

$$Z^{\gamma}(x) = \left(\log \frac{1}{x}\right)^{-\gamma} \quad x \in (0,1) \text{ and } \gamma > 0.$$

Let $T = [a,b]$ be a finite interval and consider a continuous function $K : T \to \mathbb{R}$. Denote by $\Delta^2 K(x,\tau)$ the second symmetric difference of f on J, i.e.,

$$\Delta^2 K(x,\tau) = K(x+\tau) + K(x-\tau) - 2K(x),$$

where $x \in T$, $\tau \in \left[0, \frac{|T|}{2}\right]$ and $x+\tau, x-\tau \in T$. Now we are ready to define a new class.

Definition 4.1. Let $Z_k^{1+\gamma}$, $k \in \mathbb{N}$ and $\gamma > 0$, be the set of g.i.e.m. $f : I \to I$ such that

(i) For each $\alpha \in A$ we can extend f to $\overline{I}_{\alpha}$ as an orientation preserving diffeomorphism;

(ii) On each $\overline{I}_{\alpha}$, f' has bounded variation and satisfies

$$\left\| \Delta^2 f'(\cdot,\tau) \right\|_{L^{\infty}(\overline{I}_{\alpha})} \leq C\tau Z^{\gamma}(\tau);$$

(iii) The g.i.e.m. f has k-bounded combinatorics;

(iv) The map f has genus one and has no connection.

Note that the class of real functions satisfying (ii) inequality is wider than $C^{2+\alpha}$, for any $\gamma > 0$.

We remind that the class of real functions satisfying (ii) with $Z^{\gamma}(\tau) = 1$ is called the *Zygmund class*. This class was applied to the theory of circle homeomorphisms for the first time by Hu and Sullivan (1997). Generally speaking, the function satisfying (ii) does not imply the absolute continuity of f' on $\overline{I}_{\alpha}$.

5. Main Results

We need the following notions. Let H be a non-degenerate interval, let $g : H \to \mathbb{R}$ be a diffeomorphism and let $J \subset H$ be an interval. We define the Zoom of g in H, denoted by $\Xi_H(g)$ the transformation $\Xi_H(g) = A_1 \circ g \circ A_2$ where A_1 and A_2 are orientation-preserving affine maps, which sends $[0,1]$ into H and $g(H)$ into $[0,1]$ respectively. Let M_N be a Mobius transformation $M_N : [0,1] \to [0,1]$ such that $M_N(0) = 0$, $M_N(1) = 1$ and

$$M_N(x) = \frac{xN}{1 + x(N-1)}.$$

Denote by $q_n^\alpha \in \mathbb{N}$ the first return time of the interval I_α^n to the interval I^n, i.e., $R^n(f)|_{I_\alpha^n} = f^{q_n^\alpha}$ for some $q_n^\alpha \in \mathbb{N}$. Now we define a new quantity as follows:

$$\hat{m}_n^\alpha = \exp\left(-\sum_{i=0}^{q_n^\alpha - 1} \frac{f'(d_\alpha^i) - f'(c_\alpha^i)}{2f'(d_\alpha^i)} \right),$$

where c_α^i and d_α^i are the left and right endpoints of $f^i(\overline{I}_\alpha)$, respectively. Now we are ready to formulate our main results.

Theorem 5.1. *Let $f \in Z_k^{1+\gamma}$, $\gamma \in (0,1]$. Then there exists a constant $C = C(f) > 0$ such that*

$$\left\| \Xi_{I_\alpha^n}(R^n(f)) - M_{\hat{m}_n^\alpha} \right\|_{C^1([0,1])} \le \frac{C}{n^\gamma}$$

for all $\alpha \in A$.

The sketch of proof of this theorem will be given later. Note that the class $Z_k^{1+\gamma}$ is "better" when γ increases. More precisely, if $\gamma > 1$ then second derivative of f exists on each continuity intervals of f'. This gives more opportunities to better understand behaviour of $\Xi_{I_\alpha^n}(R^n(f))$. Next, instead of $\hat{m}_n^\alpha$ we define a new quantity as follows:

$$m_n^\alpha = \exp\left(-\sum_{i=0}^{q_n^\alpha - 1} \int_{c_\alpha^i}^{d_\alpha^i} \frac{f''(x)}{2f'(x)} dx \right).$$

Note that $\log m_n^\alpha$ is called *nonlinearity* of $R^n(f)$ on I_α^n. Using differentiability of f' easily can be shown that m_n^α is exponential close to the $\hat{m}_n^\alpha$. Our second main result is the following:

Theorem 5.2. *Let* $f \in Z_k^{1+\gamma}$, $\gamma > 1$. *Then there exists a constant* $C = C(f) > 0$ *such that*

$$\left\| \Xi_{I_\alpha^n}(R^n(f)) - M_{m_n^\alpha} \right\|_{C^1([0,1])} \le \frac{C}{n^\gamma}$$

and

$$\left\| \Xi''_{I_\alpha^n}(R^n(f)) - M''_{m_n^\alpha} \right\|_{C^0([0,1])} \le \frac{C}{n^\gamma}$$

for all $\alpha \in A$.

For the nonlinearity of n-th renormalisation of f the following estimation holds:

Theorem 5.3. *Let* $f \in Z_k^{1+\gamma}$, $\gamma > 1$. *Then there exists a constant* $C = C(f) > 0$ *such that*

$$\left| m_n^\alpha - \frac{\sum_{i=0}^{q_n^\alpha - 1} \left| f^i(I_n^\alpha) \right|}{|I|} \int_0^1 \frac{f''(x)}{f'(x)} dx \right| \le \frac{C}{n^{\gamma-1}}.$$

In particular, if $\int_0^1 \frac{f''(x)}{f'(x)} dx = 0$ *then* $\left| m_n^\alpha \right| \le \frac{C}{n^{\gamma-1}}$.

Note that Theorems 5.1, 5.2 and 5.3 generalise the results of Cunha and Smania (2013).

6. Sketch of Proofs of Theorems 5.1 and 5.2

The proofs of these theorems consist from four steps.

6.1 *Step 1.*

First we analyse the set of real functions satisfying the inequality (ii) in Definition 4.1. We show that the modulus of continuity of such functions is

$$\omega(\delta) = \delta \left(\log \frac{1}{\delta} \right)^{1-\gamma}$$

if $\gamma \in (0,1)$,

$$\omega(\delta) = \delta \left(\log \log \frac{1}{\delta} \right)$$

if $\gamma = 1$, and they are differentiable if $\gamma > 1$. Moreover, we prove that if a function g satisfies inequality (ii) then it " almost" preserves *barycentres*, that is, for any interval $I = [a,b]$ we have

$$g(za+(1-z)b) = zg(a)+(1-z)g(b)+O\left(|I|Z^{\gamma}(|I|)\right)$$

where z is zoom of the interval I.

6.2 *Step 2*

We define the distortion of the interval $I=[a,b]$ with respect to the function g as follows

$$\Re(I;g) = \frac{|g(I)|}{|I|}.$$

Henceforth, take any $x \in I$ we consider the distortions:

$$\Re_a(x) = \Re([a,x];g) \text{ and } \Re_b(x) = \Re([x,b];g).$$

and we study these distortions as the functions of $x \in I$. Utilising relations in Step 1 we prove the following several estimations:

$$\frac{\Re_a(x)}{\Re_b(x)} - 1 = \frac{g(a)-g(b)}{2g(b)} + O\left(|I|Z^{\gamma}(|I|) + |g(a)-g(b)|\Omega(|I|,\gamma)\right)$$

$$(x-a)(b-x)\left(\frac{\Re'_b(x)-\Re'_a(x)}{b-a}\right) = O(|I|Z^{\gamma}(|I|))$$

for $\gamma > 0$. In the case of $\gamma > 1$ we prove that

$$(x-a)(b-x)(\Re''_a(x)-\Re''_b(x)) = O(|I|P_{\gamma}(|I|))$$

$$\Re'_b(x) = \Re'_a(x) + O(P_{\gamma}(|I|))$$

where $\Omega(\cdot,\gamma)$ is modulus of continuity of g for the different $\gamma > 0$ and $P_{\gamma}(\cdot)$ is defined by

$$P_{\gamma}(x) = \sum_{n=1}^{\infty} Z^{\gamma}(x2^{-n})$$

6.3 *Step 3*

By definition of Rauzy-Veech renormalisation it implies that the system of intervals

$$\Im_n = \left\{ f^i(I_{\alpha}^n), 0 \le i < q_n^{\alpha}, \forall \alpha \in A \right\}$$

consists a partition on $[0,1)$ that is,

$$[0,1) = \bigcup_{\alpha \in A} \bigcup_{i=0}^{q_n^{\alpha}-1} f^i(I_{\alpha}^n).$$

Denote by

$$|\Im_n| = \max_{\alpha \in A} \max_{0 \le i < q_n^{\alpha}} \left\{ |f^i(I_{\alpha}^n)| \right\}.$$

Cunha and Smania (2013) have shown that if f has k-bounded combinatorics and $\log f'$ has bounded variation then there exists a $\lambda \in (0,1)$ such that $|\Im_n| \leq \lambda^n$. Next we define

$$\wp_n^\alpha(z) = \log \frac{\Re([c_n^\alpha, x]; f^{q_n^\alpha})}{\Re([x, d_n^\alpha]; f^{q_n^\alpha})} + \log m_n^\alpha,\ x \in I_n^\alpha = [c_n^\alpha, d_n^\alpha],$$

where $z = \dfrac{x - c_n^\alpha}{d_n^\alpha - c_n^\alpha}$. Since the distortion is multiplicative with respect to composition, we have

$$\wp_n^\alpha(z) = \sum_{i=0}^{q_n^\alpha - 1} \log \frac{\Re([c_{n,i}^\alpha, x_i]; f)}{\Re([x_i, d_{n,i}^\alpha], f)} + \log m_n^\alpha$$

where $c_{n,i}^\alpha, x_i$ and $d_{n,i}^\alpha$ are i-th iteration of $c_{n,i}^\alpha, x_i$ and $d_{n,i}^\alpha$, respectively. Using this and relations in Step 2 we show that

$$\left\| \wp_n^\alpha \right\|_{C^0([0,1])} \leq \frac{C}{n^{\gamma'}},$$

$$\left\| Id(1 - Id)(\wp_n^\alpha)' \right\|_{C^0([0,1])} \leq \frac{C}{n^\gamma}$$

for $\gamma > 0$, for $\gamma > 1$ we show that

$$\left\| Id(1 - Id)(\wp_n^\alpha)'' \right\|_{C^0([0,1])} \leq \frac{C}{n^{\gamma-1}}$$

for all $\alpha \in A$.

6.4 *Step 4*

A not too hard calculation shows that for any $\alpha \in A$ we have

$$\frac{1 - \Xi_{I_\alpha^n}(R^n(f))(z)}{\Xi_{I_\alpha^n}(R^n(f))(z)} \frac{z}{1-z} = \frac{\Re([c_n^\alpha, x]; f^{q_n^\alpha})}{\Re([x, d_n^\alpha]; f^{q_n^\alpha})}. \tag{1}$$

On the other hand, from the above relation it follows

$$\frac{\Re([c_n^\alpha, x]; f^{q_n^\alpha})}{\Re([x, d_n^\alpha]; f^{q_n^\alpha})} = \frac{1}{m^\alpha} \exp(\wp_n^\alpha(z)). \tag{2}$$

The relations (1) and (2) relations give us

$$\Xi_{I_\alpha^n}(R^n(f))(z) = \frac{z m_n^\alpha}{(1-z)\exp(\wp_n^\alpha(z)) + z m_n^\alpha}. \tag{3}$$

Successively twice differentiating (3), we face to the expressions $\wp_n^{\alpha}(z), z(1-z)(\wp_n^{\alpha}(z))'$ and $(1-z)(\wp_n^{\alpha}(z))''$. Due to the estimations (1), (2) and (3) they are estimated with $O(n^{-\gamma})$ and $O(n^{-\gamma+1})$ which imply the proofs of Theorems 5.1 and 5.2.

7. Sketch of Proof of Theorem 5.3

In fact the proof of Theorem 5.3 follows closely that of Cunha and Smania (2013). Following them we introduce a certain symbolic dynamics. We study properties of *admissible cylinders*. In the estimation process of

$$\int_{I_\alpha^n} \frac{(R^n(f)(x))''}{(R^n(f)(x))'} dx$$

we face to the difference of nonlinearity of f on atoms of partition $\Im_n$. Since the norm of this partition is exponential small and modulus of continuity of nonlinearity of f is P_γ we have

$$\left| \frac{f''(x_i)}{f'(x_i)} - \frac{f''(y_i^{\beta})}{f'(y_i^{\beta})} \right| \le CP_\gamma(|\Im_n|) \le \frac{C}{n^{\gamma-1}}$$

for all i and $\beta \in A$. This finishes the proof.

8. Singularity of Conjugacy

Further, using above theorems we study the conjugacy of two g.i.e.m. f and g. Given two infinitely renormalisable g.i.e.m. f and g, defined with the same alphabet A, we say that f and g have the *same combinatorics* if $\pi(f) = \pi(g)$ and the n-th renormalisation of f and g have the same type, for every $n \in \mathbb{N}$. It follows that $\pi^n(f) = \pi^n(g)$ for every n, where $\pi^n(f)$ is the combinatorial data of the n-th renormalisation of f. The map $f : S^1 \to S^1$ is a piecewise smooth homeomorphism on the circle if f is homeomorphism, has jumps in the first derivative on finitely many points, that we call break points, and f is smooth outside its break points. The set

$$BP_f = \left\{ x \in S^1 : BP_f := \frac{Df(x-0)}{Df(x+0)} \neq 1 \right\}$$

is called the set of break points f and the number $BP_f(x)$ is called the break f at x. Denote by $BP_f = \{x_1, ..., x_m\}$ and $BP_g = \{y_1, ..., y_n\}$. We say that two piecewise smooth homeomorphisms on the circle are *break-equivalents* if there exists a topological conjugacy h such that

$$h(BP_f) = BP_g$$

and

$$BP_f(x_i) = BP_g(h(x_i)).$$

It is easy to see that if there is a C^1 conjugacy between f and g then f and g are break-equivalents.

Theorem 8.1. *Let* $f, g \in Z_k^{1+\gamma}$ *be such that*

i. f and g have the same combinatorics;
ii. f and g are not break-equivalents;

then the conjugating map h *between* f *and* g *is singular.*

This theorem extends the result of Cunha and Smania (2014).

9. Sketch of Proof of Theorem 8.1

The main approach for proving this theorem is to study the behaviour of sequence

$$\left\{ \log \frac{(R_{SK}^n(g)(h(x)))'}{(R_{SK}^n(f)(x))'} \right\}_{n=1,2,\ldots}$$

and

$$\left\{ \log \frac{(R_{MS}^n(g)(h(x)))'}{(R_{MS}^n(f)(x))'} \right\}_{n=1,2,\ldots}$$

where R_{SK} and R_{MS} R_{MS} are Sinai-Khanin and Mackay-Stark renormalisations respectively. Similar argument has been used by Herman (1979) for investigating conjugations between piecewise linear circle homeomorphisms with two break points and linear rotation. Denote by $\left(L_{SK}^n, M_{SK}^n\right)$ and $\left(L_{MS}^n, M_{MS}^n\right)$ n-th *commuting pairs* of Sinai-Khanin and Mackay-Stark renormalisations respectively. First we give a necessary condition for absolute continuity of conjugation as follows:

Let f and g have the same combinatorics. If the conjugation map h between f and g is absolute continuous then for all $\delta > 0$

$$\lim_{n\to\infty} l\left(x : \left|\log(L_{SK}^n(g)(h(x)))' - \log(L_{SK}^n(f)(x))'\right| > \delta\right) = 0$$

and

$$\lim_{n\to\infty} l\left(x : \left|\log(M_{SK}^n(g)(h(x)))' - \log(M_{SK}^n(f)(x))'\right| > \delta\right) = 0.$$

Similar result is true for Mackay-Stark's commuting pair under assumption of k-bounded combinatorics. Next we show that for any $\alpha \in A$ there exist the universal $\delta_0 > 0$ and U_α^n such that $U_\alpha^n \subset I_\alpha^n$ and

$$\left|\log(L_{SK}^n(g)(h(x)))' - \log(L_{SK}^n(f)(x))'\right| > \delta_0$$

and

$$\left|\log(M_{SK}^n(g)(h(x)))' - \log(M_{SK}^n(f)(x))'\right| > \delta_0$$

on the set

$$x \in \bigcup_{i=0}^{k_n^\alpha} f^i(U_\alpha^n)$$

where $k_n^\alpha \le q_n^\alpha$. Finally we show that, the Lebesgue measure of the set

$$\bigcup_{\alpha \in A} \bigcup_{i=0}^{k_n^\alpha - 1} f^i(U_\alpha^n)$$

cannot tends to zero for sufficiently large n. This contradicts to the above necessary condition for absolute continuity of conjugation.

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