CLOSE-TO-CONVEX FUNCTIONS WITH STARLIKE POWERS
(Fungsi Hampir Cembung dengan Kuasa Bak Bintang)

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ABSTRACT

Let $K^\alpha$ be the set of functions $f$, analytic in $z \in D = \{z : |z| < 1\}$ satisfying
\[
\text{Re} \left( \frac{f'(z)}{(g(z)/z)^\alpha} \right) > 0, \text{ for } g \text{ starlike in } D \text{ and } 0 \leq \alpha \leq 1.
\]
It is shown that such functions form a subset of the close-to-convex functions. Sharp bounds for the coefficients are given and the Fekete-Szegö problem is solved.

Keywords: univalent functions; starlike functions; close-to-convex functions; coefficients; Fekete-Szegö

1. Introduction

Let $S$ be the class of analytic normalised univalent functions $f$ defined in $z \in D = \{z : |z| < 1\}$ and given by
\[
f(z) = z + \sum_{n=2}^\infty a_n z^n.
\]

Denote by $S^*$ the subset of functions, starlike with respect to the origin and by $K$ the subset of close-to-convex functions. Then $f \in S^*$ if and only if, for $z \in D = \{z : |z| < 1\}$,
\[
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > 0
\]
and $f \in K$ if and only if, there exists $g \in S^*$ such that
\[
\text{Re} \left( \frac{zf''(z)}{g(z)} \right) > 0.
\]
Hence $S^* \subset K \subset S$.

For $0 \leq \alpha \leq 1$, the subset $B(\alpha)$ of Bazilevič functions (1955) has been widely studied and is defined for $z \in D = \{z : |z| < 1\}$ by

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)^{1-\alpha}g(z)^{\alpha}} \right\} > 0$$

for some $g \in S^*$.

In the original paper of Bazilevič (1995), it was shown that $B(\alpha) \subset S$. Taking $g(z) \equiv z$ gives the class $B_1(\alpha)$, which has been extensively studied e.g. Singh (1973), Thomas (1985) and Thomas (1991). We note that $B_1(0)$ is the well-known class $R$ of functions whose derivative has positive real part in $D$.

Thus $f \in B_1(\alpha)$, if and only if

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)^{1-\alpha}z^{\alpha}} \right\} > 0.$$

We choose a different route as follows.

2. Main Result

**Definition 1.** For $0 \leq \alpha \leq 1$, denote by $K^\alpha$, the set of functions $f$ analytic in $z \in D = \{z : |z| < 1\}$ and given by (1) such that for some $g \in S^*$,

$$\text{Re} \left\{ \frac{zf'(z)}{z^{1-\alpha}g(z)^{\alpha}} \right\} > 0,$$

which is equivalent to

$$\text{Re} \left\{ \frac{f'(z)}{(g(z)/z)^{\alpha}} \right\} > 0.$$

**Theorem 1.** Let $f \in K^\alpha$. Then $f \in K$ and so is univalent for $z \in D = \{z : |z| < 1\}$.

**Proof.** Let $G(z) = z(g(z)/z)^{\alpha}$. Then it is easily seen that $G \in S^*(1 - \alpha)$ for $0 < \alpha \leq 1$, where $S^*(1 - \alpha)$ is the class of functions starlike of order $1 - \alpha$. Thus $K^\alpha \subset K$ and so functions in $K^\alpha$ are univalent. □
Theorem 2. Let \( f \in K^\alpha \) and be given by (1) and \( F \) be defined for \( z \in D = \{z : |z| < 1\} \) by

\[
F'(z) = 1 + \sum_{n=1}^{\infty} n \gamma_n(\alpha) z^n = \left(\frac{1+z}{1-z} \right)^{2\alpha+1} \left(1 + \sum_{k=1}^{\infty} \left(\frac{2\alpha + k - 1}{k}\right) z^k \right) \left(1 + 2 \sum_{k=1}^{\infty} z^k \right)
\]

(2)

with \( \gamma_1(\alpha) = 1 \). Then \( |a_n| \leq \gamma_n(\alpha) \) for \( n \geq 2 \), where \( \gamma_n(\alpha) = \frac{2n^{2\alpha-1}}{\Gamma(2\alpha+1)} \) as \( n \to \infty \).

The inequality is sharp.

Proof: Write

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad \left(\frac{g(z)}{z}\right)^n = 1 + \sum_{k=1}^{\infty} B_k(\alpha) z^k \quad \text{and} \quad h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k,
\]

where \( h \in P\), the class of functions whose derivative has positive real part in \( D \). Then

\[
f''(z) = \left(\frac{g(z)}{z}\right)^n h(z) = \left(1 + \sum_{k=1}^{\infty} B_k(\alpha) z^k \right) \left(1 + \sum_{k=1}^{\infty} c_k z^k \right).
\]

(3)

Equating the coefficients, we obtain for \( n \geq 2 \), \( na_n = \sum_{i=0}^{n-1} B_i(\alpha)c_{n-i} \), where \( B_0(\alpha) = c_0 = 1 \).

A result of Klein (1968) shows that for \( g \in S^* \) and \( i \geq 1 \),

\[
|B_i(\alpha)| \leq \left|\begin{array}{c}
-2\alpha \\
i
\end{array}\right| = \left(\frac{2\alpha + i - 1}{i}\right)
\]

and so using the well-known inequality \( |c_n| \leq 2 \) for \( h \in P \) and comparing the coefficients in (2) and (3), the result follows. \( \square \)

We note that elementary analysis shows that \( \gamma_n(\alpha) = \frac{2n^{2\alpha-1}}{\Gamma(2\alpha+1)} \) as \( n \to \infty \).

Theorem 1 gives the inequalities \( |a_2| \leq 1 + \alpha \) and \( |a_3| \leq \frac{1}{3} \left(2\alpha^2 + 5\alpha + 2\right) \), which we will use in the following theorem.

We now solve the Fekete-Szegö problem for functions in \( K^\alpha \), noting that when \( \alpha = 1 \) we obtain the classical result of Keogh and Merkes (1969).
Theorem 3. Let $f \in K^{\alpha}$. Then

$$|a_{j} - \mu a_{j}^{2}| \leq \begin{cases} \frac{1}{3}(2 + \alpha)(1 + 2\alpha) - \mu(1 + \alpha)^{2} & \text{if } \mu \leq \frac{2\alpha}{3(1 + \alpha)}, \\ \frac{\alpha}{3} + \frac{2}{3}(1 - \alpha^{2}) + \frac{4\alpha^{2}}{9\mu} & \text{if } \frac{2\alpha}{3(1 + \alpha)} \leq \mu \leq \frac{2}{3}, \\ \frac{2}{3} & \text{if } \frac{2}{3} \leq \mu \leq \frac{2(2 + \alpha)}{3(1 + \alpha)}, \\ \mu(1 + \alpha^{2}) - \frac{1}{3}(2 + \alpha)(1 + 2\alpha) & \text{if } \mu \geq \frac{2(2 + \alpha)}{3(1 + \alpha)}. \end{cases}$$

All inequalities are sharp.

Proof: We first recall the Fekete-Szegö inequality for starlike functions (see Keogh and Merkes (1969), for example), which states that for $g$ starlike in $D$,

$$|b_{3} - \mu b_{2}^{2}| \leq \text{Max } \{1, 3 - 4\mu\},$$

where $\mu$ is any real number and where, $b_{2}$ and $b_{3}$ are coefficients of the starlike function $g$. Equating coefficients in (3), gives

$$|a_{j} - \mu a_{j}^{2}| = \frac{\alpha}{3} b_{j} - \frac{\alpha}{12}(2(1 - \alpha) + 3\alpha\mu)b_{j}^{2} + \frac{\alpha}{6} b_{j} c_{j} (2 - 3\mu) + \frac{1}{3} \left( c_{j} - \frac{3\mu - 3\alpha^{2}}{4} \right).$$

$$\leq \frac{\alpha}{3} b_{j} - \frac{1}{4}(2(1 - \alpha) + 3\alpha)\left|\frac{\alpha}{6} b_{j} c_{j} (2 - 3\mu)\right| + \frac{1}{3}\left|\frac{3\mu - 3\alpha^{2}}{4}\right| c_{j}^{2} + \frac{1}{3}\left| c_{j} - \frac{1}{2} c_{j}^{2} \right| + \frac{1}{12} |2 - 3\mu||c_{j}|^{2}. \quad (4)$$

(i) The case $\frac{2\alpha}{3(1 + \alpha)} \leq \mu \leq \frac{2}{3}$.

If $0 \leq \mu \leq \frac{2}{3}$, then $|3\alpha\mu - 2\alpha - 1| \geq 1$ and so using the well-known bound $\left| c_{j} - \frac{1}{2} c_{j}^{2} \right| \leq 2 - \frac{1}{2} |c_{j}|^{2}$ for $h \in P$, (4) gives

$$|a_{j} - \mu a_{j}^{2}| \leq \frac{\alpha}{3} |3\alpha\mu - 2\alpha - 1| + \frac{\alpha}{3} |2 - 3\mu||c_{j}| + \frac{1}{3} \left( 2 - \frac{1}{2} |c_{j}|^{2} \right) + \frac{1}{12} (2 - 3\mu)|c_{j}|^{2}.$$

Since $3\alpha\mu - 2\alpha - 1 \leq 0$, when $0 \leq \mu \leq \frac{2}{3}$, we have
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\[ |a_1 - \mu a_1^2| \leq \frac{\alpha}{3} (1 + 2\alpha - 3\alpha \mu) + \frac{\alpha}{3} (2 - 3\mu)|c_1| + \frac{1}{3} \left( 2 - \frac{1}{2}|c_1|^2 \right) + \frac{1}{12} (2 - 3\mu)|c_1|^2 \]

\[ = \frac{\alpha}{3} (1 + 2\alpha - 3\alpha \mu) + \frac{\alpha}{3} (2 - 3\mu)|c_1| + \frac{2}{3} - \frac{\mu}{4}|c_1|^2 \]

\[ = \Phi(x) \text{ say,} \]

with \(|c_1| = x\). It is easy to see that \(\Phi(x)\) is maximum at \(x_0 = \frac{2\alpha}{3\mu}(2 - 3\mu)\) and

\[ \Phi(x_0) = \frac{\alpha}{3} + \frac{2}{3}(1 - \alpha^2) + \frac{4\alpha^2}{9\mu}, \]

but since \(x_0 \leq 2\), we have \(\mu \geq \frac{2\alpha}{3(1 + \alpha)}\). Equality is attained when \(c_1 = \frac{2\alpha}{3\mu}(2 - 3\mu)\), \(c_2 = b_2 = 2\) and \(b_1 = 3\).

(ii) The case \(\mu \leq \frac{2\alpha}{3(1 + \alpha)}\).

First note that the above shows that when \(\mu = \frac{2\alpha}{3(1 + \alpha)}\),

\[ \left| a_1 - \frac{2\alpha}{3(1 + \alpha)}a_1^2 \right| \leq \frac{2 + 3\alpha}{3}, \]

and so writing

\[ a_1 - \mu a_1^2 = \frac{3\mu(1 + \alpha)}{2\alpha} \left( a_1 - \frac{2\alpha}{3(1 + \alpha)}a_1^2 \right) + \left( 1 - \frac{3\mu(1 + \alpha)}{2\alpha} \right) a_1, \]

and using the bound \(|a_1| \leq \frac{1}{3}(2 + \alpha)(1 + 2\alpha)\) obtained from Theorem 1, we have

\[ |a_1 - \mu a_1^2| \leq \frac{1}{3}(2 + \alpha)(1 + 2\alpha) - \mu(1 + \alpha)^2. \]

Equality is attained when \(b_1 = 3\), \(b_2 = c_1 = c_2 = 2\).

(iii) The case \(\frac{2}{3} \leq \mu \leq \frac{2(2 + \alpha)}{3(1 + \alpha)}\).

Since \(\text{Max}\{1, |3\alpha \mu - 2\alpha - 1|\} = 1\) on this interval,
\[
|a_i - \mu a_i^2| \leq \alpha + \frac{2\alpha}{3}(3\mu - 2)|c_i| + \frac{1}{3}\left(2 - \frac{1}{2}|c_i|^2\right) + \frac{1}{12}(3\mu - 2)|c_i|^2
\]

\[
= \alpha + \frac{2\alpha}{3}(3\mu - 2)x + \frac{1}{3}\left(2 - \frac{1}{2}x^2\right) + \frac{1}{12}(3\mu - 2)x^2
\]

\[
= \Psi(x) \text{ say,}
\]

where again \(x = |c_i|\). Since \(\Psi'(x) = 0\) at \(x_0 = \frac{2\alpha(3\mu - 2)}{4 - 3\mu}\), and \(x_0 \leq 2\), it follows that \(\mu \leq \frac{2(2 + \alpha)}{3(1 + \alpha)}\). We finally note that \(\Psi(x_0) = \frac{2 + \alpha}{3}\) and so the proof for \(\frac{2}{3} \leq \mu \leq \frac{2(2 + \alpha)}{3(1 + \alpha)}\) is complete. Equality is attained when \(c_i = b_2 = 0\), \(c_2 = 2\) and \(b_j = 1\).

(iv) The case \(\mu \geq \frac{2(2 + \alpha)}{3(1 + \alpha)}\).

Writing \(a_i - \mu a_i^2 = a_i - \frac{2(2 + \alpha)}{3(1 + \alpha)} a_i^2 + \left(\frac{2(2 + \alpha)}{3(1 + \alpha)} - \mu\right) a_i^2\),

\[
|a_i - \mu a_i^2| \leq \left|a_i - \frac{2(2 + \alpha)}{3(1 + \alpha)} a_i^2\right| + \left(\mu - \frac{2(2 + \alpha)}{3(1 + \alpha)}\right)^2 a_i^2
\]

\[
\leq \frac{2 + \alpha}{3} + \left(\mu - \frac{2(2 + \alpha)}{3(1 + \alpha)}\right)^2 (1 + \alpha)^2 = \mu(1 + \alpha)^3 - \frac{1}{3}(2 + \alpha)(1 + 2\alpha).
\]

where we have used the bound \(|a_i| \leq 1 + \alpha\) obtained from Theorem 1. Choosing \(b_2 = -2i\), \(b_3 = -3\), \(c_1 = 2i\) and \(c_2 = -2\) shows that the inequality is sharp. \(\square\)

We next define a related class \(C^\alpha\).

**Definition 2.** For \(0 \leq \alpha \leq 1\), denote by \(C^\alpha\), the set of function \(f\) analytic in \(z \in D = \{z:|z| < 1\}\) and given by (1) such that for some \(g \in S^\alpha\),

\[
\Re\left\{z(f'(z)) \over z^{1-\alpha}g(z)\right\} > 0,
\]

which is equivalent to

\[
\Re\left\{z(f'(z)) \over (g(z)/z)^\alpha\right\} > 0
\]
and so \( f \in C^\alpha \) if and only if, \( zf' \in K^\alpha \).

Using the same methods as in the case of \( K^\alpha \) it is easily established that \( C^\alpha \subset C^\nu \), the set of quasi-convex functions first studied by Noor and Thomas (1980), and since \( C^\nu \subset K^\nu \), functions in \( C^\alpha \) are also close-to-convex and hence univalent.

We note that since \( f \in C^\alpha \) if and only if, \( zf' \in K^\alpha \), the coefficients of functions in \( C^\alpha \) satisfy \( n|a_n| \leq \gamma_n(\alpha) \) for \( n \geq 2 \).

Similar techniques as those employed in Theorem 3 gives the following Fekete-Szegö theorem, the proof of which we omit.

**Theorem 4.** Let \( f \in C^\alpha \). Then

\[
\left| a_3 - \mu a_2^2 \right| \leq \begin{cases} 
\frac{1}{9} (2 + \alpha)(1 + 2\alpha) - \frac{\mu}{4} (1 + \alpha)^2 & \text{if } \mu \leq \frac{8\alpha}{9(1 + \alpha)}, \\
\frac{\alpha}{3} + \frac{2}{3} (1 - \alpha^2) + \frac{16\alpha^2}{81\mu} & \text{if } \frac{8\alpha}{9(1 + \alpha)} \leq \mu \leq \frac{1}{9}, \\
\frac{2 + \alpha}{3} & \text{if } \frac{8}{9} \leq \mu \leq \frac{8(2 + \alpha)}{9(1 + \alpha)}, \\
\mu(1 + \alpha^2) - \frac{1}{3} (2 + \alpha)(1 + 2\alpha) & \text{if } \mu \geq \frac{8(2 + \alpha)}{9(1 + \alpha)}. 
\end{cases}
\]

All inequalities are sharp.

**References**


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