Journal of Quality Measurement and Analysis *JQMA* 11(2) 2015, 11-16
Jurnal Pengukuran Kualiti dan Analisis

ON THE SECOND HANKEL DETERMINANT OF SOME ANALYTIC FUNCTIONS

(Mengenai Penentu Hankel Kedua bagi Beberapa Fungsi Analisis)

D. K. THOMAS & SARIKA VERMA

ABSTRACT

Let the function f be analytic in $z \in D = \{z : |z| < 1\}$ and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For $0 \le \alpha \le 1$, denote by $V(\alpha)$ and $U(\alpha)$, the sets of functions analytic in D, satisfying $\operatorname{Re}\left((1-\alpha) f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0$ and $\operatorname{Re}\left((1-\alpha)\frac{f(z)}{z} + \alpha\frac{zf'(z)}{f(z)}\right) > 0$ respectively, so that $f \in V(\alpha) \Leftrightarrow zf' \in U(\alpha)$. We give sharp bounds for the Hankel determinant $H_2 = |a_2 a_4 - a_3^2|$ for $f \in V(\alpha)$ and $f \in U(\alpha)$.

Keywords: univalent functions; starlike and convex functions; Hankel determinant

ABSTRAK

Andaikan fungsi f analisis dalam $z \in D = \{z : |z| < 1\}$ dan diberi oleh $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Untuk $0 \le \alpha \le 1$, set fungsi analisis dalam D yang dilambangi oleh $V(\alpha)$ dan $U(\alpha)$, masing-masing memenuhi $\operatorname{Ny}\left((1-\alpha) f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0$ dan $\operatorname{Ny}\left((1-\alpha)\frac{f(z)}{z} + \alpha\frac{zf'(z)}{f(z)}\right) > 0$ sehinggakan $f \in V(\alpha) \Leftrightarrow zf' \in U(\alpha)$. Batas terbaik bagi penentu Hankel $H_2 = |a_2 a_4 - a_3^2|$ untuk $f \in V(\alpha)$ dan $f \in U(\alpha)$ diberi.

Kata kunci: fungsi univalen; fungsi bak bintang dan cembung; penentu Hankel

1. Introduction

Let S be the class of analytic normalised univalent functions f, defined in $z \in D = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

The qth Hankel determinant of f defined for $q \ge 1$ and $n \ge 1$ as follows has been extensively studied.

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & \vdots \\ \vdots & & \\ a_{n+q-1} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Much attention has been given in the literature to finding upper bounds for the Hankel determinant whose elements are the coefficients of univalent, or multivalent functions, see e.g. (Hayman 1968; Janteng *et al.* 2007; Noonan & Thomas 1976; Pommerenke 1967). The correct order of growth for $H_q(n)$ when $f \in S$ is as yet unknown (Pommerenke 1967), whereas exact bounds have been obtained in the case $q = 2$ and $n = 2$ for a variety of subclasses of S, most of these stemming from the method used in Libera and Zlotkiewicz (1983). In this paper we give some sharp bounds for $H_2(2) = \left|a_2 a_4 - a_3^2\right|$, which unify and extend well-known results. In particular we find the exact solution to a problem posed in Verma *et al.* (2012), which sought to find the sharp upper bound of $H_2(2)$ for a set of analytic functions which unifies the starlike functions, and functions satisfying the condition $\operatorname{Re}\left(\frac{f(z)}{z}\right) > 0.$

2. Preliminaries

We recall the following subclasses of S.

Suppose that f is analytic in $z \in D = \{z : |z| < 1\}$ and given by (1). Then f is respectively starlike and convex in D if, and only if,

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,$$

and

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 .$$

We denote these classes by S^* and C respectively.

Our results will also relate to the class R of functions satisfying $\operatorname{Re} f'(z) > 0$ for $z \in D$.

For f analytic in D, let $V(\alpha)$ and $U(\alpha)$ be the sets of functions satisfying respectively

$$\operatorname{Re}\left((1-\alpha) f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0 , \tag{2}$$

and

$$\operatorname{Re}\left((1-\alpha)\frac{f(z)}{z}+\alpha\frac{zf'(z)}{f(z)}\right)>0, \tag{3}$$

so that $f\in V(\alpha)\Leftrightarrow zf'\in U(\alpha)$.

Attempts to find sharp bounds for the second Hankel determinant of classes wider than $V(\alpha)$ and $U(\alpha)$ were considered in Verma *et al.* (2012) and Zhao *et al.* (2014), but only partial answers were obtained. In this paper we give sharp results for functions in $V(\alpha)$ and $U(\alpha)$.

We shall use the following, which has been used extensively, see e.g. Libera and Zlotkiewicz (1983).

Lemma 2.1. *Suppose* $p\in P$ *, the class of functions satisfying* $\operatorname{Re}p(z)>0$ *for* $z\in D$*, with coefficients given by*

$$p(z)=1+\sum_{n=1}^{\infty}p_n z^n.$$

Then for some complex valued x with $|x|\le 1$, and some complex valued ζ with $|\zeta|\le 1$,

$$2p_2=p_1^2+x\left(4-p_1^2\right),$$
$$4p_3=p_1^3+2\left(4-p_1^2\right)p_1x-p_1\left(4-p_1^2\right)x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)\zeta.$$

Also $|p_n|\le 2$ for $n\ge 1$.

3. Main Results

Theorem 3.1. *For* $f\in V(\alpha)$ *and given by* (2),

$$H_2(2)\le\begin{cases}\dfrac{4}{9(1+a)^2} & \text{if } 0\le\alpha\le\dfrac{5}{17},\\[2ex] \dfrac{4}{9(1+\alpha)^2}+\dfrac{(5-17a)^2}{288(1+\alpha)^2(1+2\alpha)^2} & \text{if } \dfrac{5}{17}<\alpha\le 1.\end{cases}$$

The inequalities are sharp.

Proof. For $p \in P$, write

$$(1-\alpha) f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z)$$

and equate coefficients to obtain

$$a_2 = \frac{p_1}{2},$$

$$a_3 = \frac{\alpha p_1^2 + p_2}{3(1+\alpha)},$$

$$a_4 = \frac{1}{4(1+2\alpha)}\left(p_3 + \frac{\alpha(2\alpha-1)p_1^3}{1+\alpha} + \frac{3\alpha p_1 p_2}{1+\alpha}\right).$$

Thus

$$H_2(2) = \left|a_2 a_4 - a_3^2\right| = \left|\begin{array}{l} \dfrac{\alpha(-9+\alpha+2\alpha^2)p_1^4}{72(1+\alpha)^2(1+2\alpha)} + \dfrac{\alpha(11-5\alpha)p_1^2 p_2}{72(1+\alpha)^2(1+2\alpha)} - \dfrac{p_2^2}{9(1+\alpha)^2} \\ + \dfrac{p_1 p_3}{8(1+2\alpha)} \end{array}\right|.$$

We now use Lemma 2.1 to express p_2 and p_3 in terms of p_1 and, without loss in generality, normalise p_1 so that $p_1 = p$, where $0 \le p \le 2$. Using the triangle inequality then gives

$$\begin{aligned} H_2(2) \le \frac{1}{288}&\left|\frac{(1-12\alpha+3\alpha^2+8\alpha^3)p^4}{(1+\alpha)^2(1+2\alpha)}\right| + \frac{(1+13\alpha+4\alpha^2)p^2(4-p^2)|y|}{144(1+\alpha)^2(1+2\alpha)} \\ &+ \frac{p^2(4-p^2)|y|^2}{32(1+2\alpha)} + \frac{(4-p^2)^2|y|^2}{36(1+\alpha)^2} + \frac{p(4-p^2)(1-|y|^2)}{16(1+2\alpha)} := \phi(|y|). \end{aligned}$$

Elementary calculus shows that $\phi'(|y|) \ge 0$ when $0 \le |y| \le 1$, $0 \le \alpha \le 1$ and $0 \le p \le 2$. Thus $\phi(|y|) \le \phi(1)$, and so

$$H_2(2) \le \frac{1}{288}\left|\frac{\left(1-12\alpha+3\alpha^2+8\alpha^3\right)p^4}{(1+\alpha)^2(1+2\alpha)}\right| + \frac{\left(1+13\alpha+4\alpha^2\right)p^2\left(4-p^2\right)}{144(1+\alpha)^2(1+2\alpha)}$$
$$+\frac{p^2\left(4-p^2\right)}{32(1+2\alpha)}+\frac{\left(4-p^2\right)^2}{36(1+\alpha)^2}. \tag{4}$$

Thus we need to maximise (4) over $0 \le p \le 2$, for $0 \le \alpha \le 1$. We consider two cases as follows.

Case (i) $1-12\alpha+3\alpha^2+8\alpha^3 \ge 0$. Here $0 \le \alpha \le \frac{1}{16}\left(-11+3\sqrt{17}\right)$, so that (4) now gives

$$H_2(2) \le \frac{4}{9(1+a)^2}+\frac{(-5+17a)p^2}{72(1+a)(1+2a)}+\frac{\left(-1-20a-7a^2+4a^3\right)p^4}{144(1+a)^2(1+2a)}$$
$$\le \frac{4}{9(1+a)^2},$$

when $0 \le \alpha \le \frac{1}{16}\left(-11+3\sqrt{17}\right)$ and $0 \le p \le 2$.

Case (ii) $1-12\alpha+3\alpha^2+8\alpha^3 \le 0$. Here $\frac{1}{16}\left(-11+3\sqrt{17}\right) \le \alpha \le 1$, in which case (4) gives

$$H_2(2) \le \frac{4}{9(1+\alpha)^2}+\frac{(-5+17\alpha)p^2}{72(1+\alpha)(1+2\alpha)}-\frac{p^4}{72}.$$

It is now a simple exercise to show that this expression has a maximum value of $\frac{4}{9(1+\alpha)^2}$ when $\frac{1}{16}\left(-11+3\sqrt{17}\right) \le a \le \frac{5}{17}$, and a maximum value of

$$\frac{4}{9(1+\alpha)^2}+\frac{(5-17a)^2}{288(1+\alpha)^2(1+2\alpha)^2}$$

when $\frac{5}{17} < \mathrm{a} \le 1$.

Choosing $p_1 = p_2 = 0$ and $p_3 = 2$ shows that the first inequality is sharp. To see that the second inequality is sharp, choose $p_1 = p_2 = 2$ and

$$p_3 = \left(\frac{4}{1+a}\right)\left(\frac{7-4a}{9}-\frac{(5-17a)^2}{288(1+a)(1+2a)}\right).$$

noting that $|p_3| \le 2$ for $0 \le \alpha \le 1$.
This completes the proof of Theorem 3.1.

Taking $\alpha = 1$ gives the sharp estimate for the set C of convex functions (Janteng *et al.* 2007), and when $\alpha = 0$, the sharp estimate for the class R of functions whose derivative has positive real part (Janteng *et al.* 2006).

Theorem 3.2. *Let* $f \in U(\alpha)$ *and be given by* (3). *Then* $H_2(2) \le \dfrac{4}{(1+\alpha)^2}$.

The inequality is sharp.

We have included Theorem 3.2 for completeness. The relatively simple proof, which we omit, uses the same techniques as those in Theorem 3.1, and is also proved in Zhao *et al.* (2014). The case $\alpha = 0$ is contained in Mohammed and Darus (2012), and $\alpha = 1$ is the well-known result for the class of S^* of starlike functions (Janteng *et al.* 2007).

References

Hayman W.K. 1968. On the second Hankel determinant of mean univalent functions. *Proc. Lond. Math. Soc.* 3(18): 77–94.

Janteng A., Halim S. A. & Darus M. 2006. Coefficient inequality for a function whose derivative has a positive real part. *Journal of Inequalities in Pure and Applied Mathematics* **7**(2), article50.

Janteng A., Halim S. & Darus M. 2007. Hankel determinants for starlike and convex functions. *Int. J. Math. Analysis* **1(**13): 619–625.

Libera R. J. & Zlotkiewicz E. J. 1983. Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$. *Proc. Amer. Math. Soc.* **87**(2): 251–257.

Mohammed A. & Darus M. 2012. Second Hankel determinant for a class of analytic functions defined by a linear operator. *Tamkang Journal of Mathematics* **43**(3): 455–462.

Noonan J. W. & Thomas D. K. 1976. On the second Hankel determinant of a really mean *p*-valent functions. *Trans. Amer. Math. Soc*. **223**(2): 337-346.

Pommerenke Ch. 1967. On the Hankel determinants of univalent functions. *Mathematika (London).* **16(**13): 108–112.

Verma S., Gupta S. & Singh S. 2012. Bounds of Hankel determinant for a class of univalent functions. *International Journal of Mathematics and Mathematical Sciences*. Article ID 147842, 6, doi:10.1155/2012/147842.

Zhao Y.-C., Burenmandula, Li S. & Chang L. 2014. Bounds of the second Hankel determinant for two new classes of analytic functions. *Int. J. of Mathematical Sciences & Applications* **4**(1): 109–119.

Department of Mathematics
Swansea University
Singleton Park
Swansea SA2 8PP
UNITED KINGDOM
*E-mail: d.k.thomas@swansea.ac.uk**

Department of Mathematics
Sant Longowal Institute of Engineering and Technology
Punjab, Longowal 148106
INDIA
E-mail: sarika.16984@gmail.com

*Corresponding author