ON THE SECOND HANKEL DETERMINANT OF SOME ANALYTIC FUNCTIONS
(Mengenai Penentu Hankel Kedua bagi Beberapa Fungsi Analisis)

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ABSTRACT

Let the function \( f \) be analytic in \( z \in D = \{ z : |z| < 1 \} \) and be given by \( f(z) = z + \sum_{n=2}^\infty a_n z^n \).

For \( 0 \leq \alpha \leq 1 \), denote by \( V(\alpha) \) and \( U(\alpha) \), the sets of functions analytic in \( D \), satisfying
\[
\text{Re}\left( (1-\alpha)f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right) > 0 \quad \text{and} \quad \text{Re}\left( \frac{f(z)}{z} + \alpha \frac{zf''(z)}{f'(z)} \right) > 0
\]
respectively, so that \( f \in V(\alpha) \Leftrightarrow zf'' \in U(\alpha) \). We give sharp bounds for the Hankel determinant
\[
H_2 = |a_3 a_1 - a_5^2|
\]
for \( f \in V(\alpha) \) and \( f \in U(\alpha) \).

Keywords: univalent functions; starlike and convex functions; Hankel determinant

1. Introduction

Let \( S \) be the class of analytic normalised univalent functions \( f \), defined in \( z \in D = \{ z : |z| < 1 \} \) and given by
\[
f(z) = z + \sum_{n=2}^\infty a_n z^n.
\] (1)

The \( q \)th Hankel determinant of \( f \) defined for \( q \geq 1 \) and \( n \geq 1 \) as follows has been extensively studied.
Much attention has been given in the literature to finding upper bounds for the Hankel determinant whose elements are the coefficients of univalent, or multivalent functions, see e.g. (Hayman 1968; Janteng et al. 2007; Noonan & Thomas 1976; Pommerenke 1967). The correct order of growth for $H_q(n)$ when $f \in S$ is as yet unknown (Pommerenke 1967), whereas exact bounds have been obtained in the case $q = 2$ and $n = 2$ for a variety of subclasses of $S$, most of these stemming from the method used in Libera and Zlotkiewicz (1983). In this paper we give some sharp bounds for $H_2(2)$, which unify and extend well-known results. In particular we find the exact solution to a problem posed in Verma et al. (2012), which sought to find the sharp upper bound of $H_2(2)$ for a set of analytic functions which unifies the starlike functions, and functions satisfying the condition $\text{Re} \left( \frac{f(z)}{z} \right) > 0$.

2. Preliminaries
We recall the following subclasses of $S$.

Suppose that $f$ is analytic in $z \in D = \{ z : |z| < 1 \}$ and given by (1). Then $f$ is respectively starlike and convex in $D$ if, and only if,

$$ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0, $$

and

$$ \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0. $$

We denote these classes by $S^*$ and $C$ respectively.

Our results will also relate to the class $R$ of functions satisfying $\text{Re} f'(z) > 0$ for $z \in D$.

For $f$ analytic in $D$, let $V(\alpha)$ and $U(\alpha)$ be the sets of functions satisfying respectively

$$ \text{Re} \left( (1-\alpha)f'(z) + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right) > 0, \quad (2) $$

and
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\[ \text{Re} \left( (1 - \alpha) \frac{f(z)}{z} + \alpha \frac{zf''(z)}{f(z)} \right) > 0, \]

(3)

so that \( f \in V(\alpha) \Leftrightarrow zf' \in U(\alpha). \)

Attempts to find sharp bounds for the second Hankel determinant of classes wider than \( V(\alpha) \) and \( U(\alpha) \) were considered in Verma et al. (2012) and Zhao et al. (2014), but only partial answers were obtained. In this paper we give sharp results for functions in \( V(\alpha) \) and \( U(\alpha) \).

We shall use the following, which has been used extensively, see e.g. Libera and Zlotkiewicz (1983).

**Lemma 2.1.** Suppose \( p \in P \), the class of functions satisfying \( \text{Re} p(z) > 0 \) for \( z \in D \), with coefficients given by

\[ p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \]

Then for some complex valued \( x \) with \(|x| \leq 1\), and some complex valued \( \zeta \) with \(|\zeta| \leq 1\),

\[ 2p_2 = p_1^2 + x(4 - p_1^2), \]
\[ 4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)\zeta. \]

Also \(|p_n| \leq 2\) for \( n \geq 1\).

3. Main Results

**Theorem 3.1.** For \( f \in V(\alpha) \) and given by (2),

\[ H_2(2) \leq \begin{cases} \frac{4}{9(1 + a)} & \text{if } 0 \leq \alpha \leq \frac{5}{17}, \\ \frac{4}{9(1 + \alpha)^2 + \frac{(5 - 17a)^2}{288(1 + \alpha)^2(1 + 2\alpha)^2}} & \text{if } \frac{5}{17} < \alpha \leq 1. \end{cases} \]

The inequalities are sharp.
Proof. For \( p \in P \), write

\[
(1 - \alpha) f'(z) + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z)
\]

and equate coefficients to obtain

\[
a_z = \frac{p_1}{2},
\]

\[
a_3 = \frac{\alpha p_1^2 + p_2}{3(1 + \alpha)},
\]

\[
a_4 = \frac{1}{4(1 + 2\alpha)} \left( p_1 + \frac{\alpha(2\alpha - 1)p_1^3}{1 + \alpha} + \frac{3\alpha p_1 p_2}{1 + \alpha} \right).
\]

Thus

\[
H_z(2) = \left| a_z a_4 - a_3^2 \right| = \frac{\alpha((-9 + \alpha + 2\alpha^2)p_1^4)}{72(1 + \alpha)^2 (1 + 2\alpha)} + \frac{\alpha(11 - 5\alpha)p_1^2 p_2}{72(1 + \alpha)^2 (1 + 2\alpha)} - \frac{p_2^2}{9(1 + \alpha)^2} + \frac{p_1 p_4}{8(1 + 2\alpha)}
\]

We now use Lemma 2.1 to express \( p_2 \) and \( p_3 \) in terms of \( p_1 \) and, without loss in generality, normalise \( p_1 \) so that \( p_1 = p \), where \( 0 \leq p \leq 2 \). Using the triangle inequality then gives

\[
H_z(2) \leq \frac{1}{288} \left[ (-12\alpha + 3\alpha^2 + 8\alpha^3) p_1^4 + \frac{(1 + 13\alpha + 4\alpha^2)p_1^2 (4 - p^2)}{144(1 + \alpha)^2 (1 + 2\alpha)} + \frac{p_2^2 (4 - p^2)}{32(1 + 2\alpha)} + \frac{(4 - p^2)^2 |y|^2}{36(1 + \alpha)^2} + \frac{p(4 - p^2)(1 - |y|^2)}{16(1 + 2\alpha)} \right] \phi(|y|)
\]

Elementary calculus shows that \( \phi'(|y|) \geq 0 \) when \( 0 \leq |y| \leq 1 \), \( 0 \leq \alpha \leq 1 \) and \( 0 \leq p \leq 2 \). Thus \( \phi(|y|) \leq \phi(1) \), and so
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\[ H_z(2) \leq \frac{1}{288} \left( \frac{1-12\alpha + 3\alpha^2 + 8\alpha^3}{(1+\alpha)^2(1+2\alpha)} \right) p^4 + \frac{1}{144(1+\alpha)^2(1+2\alpha)} \left( \frac{1+13\alpha + 4\alpha^2}{(1+\alpha)^2(1+2\alpha)} \right) \left( 4 - p^2 \right) \]
\[ + \frac{p^2(4 - p^2)}{32(1+2\alpha)} + \frac{(4 - p^2)^2}{36(1+\alpha)^2}. \]

Thus we need to maximise (4) over \(0 \leq p \leq 2\), for \(0 \leq \alpha \leq 1\). We consider two cases as follows.

Case (i) \(1-12\alpha + 3\alpha^2 + 8\alpha^3 \geq 0\). Here \(0 \leq \alpha \leq \frac{1}{16}\left(-11+3\sqrt{17}\right)\), so that (4) now gives

\[ H_z(2) \leq \frac{4}{9(1+\alpha)^2} + \frac{(-5+17\alpha)}{72(1+\alpha)(1+2\alpha)} p^2 + \frac{(1-20a-7a^2+4a^3)}{144(1+\alpha)^2(1+2a)} p^4 \]
\[ \leq \frac{4}{9(1+\alpha)^2}, \]
when \(0 \leq \alpha \leq \frac{1}{16}\left(-11+3\sqrt{17}\right)\) and \(0 \leq p \leq 2\).

Case (ii) \(1-12\alpha + 3\alpha^2 + 8\alpha^3 \leq 0\). Here \(\frac{1}{16}\left(-11+3\sqrt{17}\right) \leq \alpha \leq 1\), in which case (4) gives

\[ H_z(2) \leq \frac{4}{9(1+\alpha)^2} + \frac{(-5+17\alpha)}{72(1+\alpha)(1+2\alpha)} p^2 + \frac{p^4}{72}. \]

It is now a simple exercise to show that this expression has a maximum value of \(\frac{4}{9(1+\alpha)^2}\) when \(\frac{1}{16}\left(-11+3\sqrt{17}\right) \leq a \leq \frac{5}{17}\), and a maximum value of

\[ \frac{4}{9(1+\alpha)^2} + \frac{(5-17\alpha)^2}{288(1+\alpha)^2(1+2\alpha)^2} \]
when \(\frac{5}{17} \leq a \leq 1\).

Choosing \(p_1 = p_2 = 0\) and \(p_3 = 2\) shows that the first inequality is sharp. To see that the second inequality is sharp, choose \(p_1 = p_2 = 2\) and

\[ p_3 = \left( \frac{4}{1+a} \right) \left( \frac{7-4a}{9} - \frac{(5-17\alpha)^2}{288(1+a)(1+2a)} \right). \]
noting that $|p_3| \leq 2$ for $0 \leq \alpha \leq 1$.

This completes the proof of Theorem 3.1.

Taking $\alpha = 1$ gives the sharp estimate for the set $C$ of convex functions (Janteng et al. 2007), and when $\alpha = 0$, the sharp estimate for the class $R$ of functions whose derivative has positive real part (Janteng et al. 2006).

**Theorem 3.2.** Let $f \in U(\alpha)$ and be given by (3). Then $H_2(2) \leq \frac{4}{(1+\alpha)^2}$.

The inequality is sharp.

We have included Theorem 3.2 for completeness. The relatively simple proof, which we omit, uses the same techniques as those in Theorem 3.1, and is also proved in Zhao et al. (2014). The case $\alpha = 0$ is contained in Mohammed and Darus (2012), and $\alpha = 1$ is the well-known result for the class of $S^*$ of starlike functions (Janteng et al. 2007).

**References**


